Estimation of Computational Error by Higher Order Approximation in the Multipoint Meshless FDM

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Introduction

- Error analysis – *essential* part of b.v. problems solution
- On Multipoint meshless FDM
  - solution *approach*
- Bases of meshless finite difference method (MFDM) *error analysis*
- Application of Multipoint method to *reference solution* generation
- *Numerical* analysis of *benchmark* problems
- Final remarks

**Objective of this research**
Investigation of the *Multipoint* MFDM application to
*aposteriori* *error analysis* by means of generation
of *high* quality HO *reference* solutions
Raising order of local MFDM approximation

ON ERROR ANALYSIS

- **Solutions**
  - true \( u^T \)
  - rough (low order, coarse mesh) \( u^L \)
  - higher quality (higher order, fine mesh) \( u^H \)

- **Generation of reference solution** \( u^H \)
  - mesh density increase \( (h) \)
  - raising approximation order \( (p) \)
  - mixture of both \( (hp) \)

- **Raising order of local MFDM approximation**
  - Defect (deferred) correction – use of MFD stars with increased number of nodes
  - Use of (additional) generalised d.o.f.
  - **Multipoint approach**
    - Use of higher order (HO) correction terms
    - \( p- \) and \( p/h- \) adaptive approach
Idea of Multipoint approach

- Given PDE (ODE)
  \[ \mathcal{L}u = f, \quad u = u(P) \]
- FD discretization
  - Standard
    \[ \mathcal{L}u_i \approx L u_i = \sum_{j(i)} C_{ij} u_j = f_i \quad \Rightarrow \quad L u_i = f_i \quad u_j = u(P_j) \]
  - Multipoint
    \[ \mathcal{L}u_i \approx \sum_{j(i)} C_{ij} u_j = \sum_{j(i)} \alpha_{ij} f_j \quad \Rightarrow \quad \mathcal{L}u_i = M f_i \]

\( \mathcal{L} \) – differential operator. In general it may be referred to: differential eqs, boundary conditions, integrand in global formulation of the considered b.v. problem

\( f_i \) – value of the whole operator \( \mathcal{L} u_i \) or its part only, e.g. a specific derivative \( u^{(k)} \)
Multipoint – simple examples

**Classic FD**

\[ Au_{i-1} + Bu_i + Cu_{i+1} = f_i \]

1st derivative

\[ u_i' \approx \frac{u_{i+1} - u_{i-1}}{2h} + O(h^2) \]

2nd derivative

\[ u_i'' \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + O(h^2) \]

**Multipoint approach**

\[ Au_{i-1} + Bu_i + Cu_{i+1} = \alpha f_{i-1} + \beta f_i + \gamma f_{i+1} \]

1st derivative

\[ \frac{u_{i+1} - u_{i-1}}{2h} \approx \frac{(u_i' + 4u_i' + u_i')}{6} + O(h^4) \]

2nd derivative

\[ \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \approx \frac{(u_i'' + 10u_i'' + u_i'')}{12} + O(h^4) \]

**Higher order approximation without raising the number of nodes!**
Generalization of the classic Multipoint FDM

- **Mesh**
  - *regular*
  - *no mesh, irregular* or *regular cloud of nodes*

- **Formulation**
  - *local*
  - *local, global, mixed*

- **Approximation**
  - *interpolation*
  - *MWLS approximation*

**Method variants**

*specific*

\[ \sum_{j(i)} C_{j} u_{j} = \sum_{j(i)} \alpha_{j} f_{j} \]

*general (variants)*

\[ \sum_{j(i)} C_{j} u_{j} = \sum_{j(i)} \alpha_{j} u^{(k)}_{j} \]
Multipoint Meshless FDM approach

new Multipoint Meshless FDM =

= Meshless FDM + Formulations +
+ MWLS approximation +
+ Collatz multipoint approach

provides higher order approximation used for

- Solution of b.v. problems
- A posteriori error analysis
Basic Multipoint formulations

MFD star and d.o.f.

\[ \sum_{j(i)} C_{ij} u_j = f_i \]

\[ \sum_{j(i)} C_{ij} u_j = \sum_{j(i)} \alpha_j f_j \]

\[ \sum_{j(i)} C_{ij} u_j = \sum_{j(i)} \alpha_j u_j^{(k)} \]

a) Standard meshless FDM  
b) Specific multipoint  
c) General multipoint
Comparison of solution convergence

- MFDM
- HO Multipoint
- General multipoint
- Specific (linear) multipoint
- Standard low order FD solution

Test 1: Series of regular meshes
True error
Error analysis – bases

• Solution types
  - $u^T$ – true solution (unknown)
  - $u^L$ – rough solution (known)
  - $u^H$ – higher order improved solution usually assumed as the reference one

Error

• Types
  - a priori, a posteriori (to examine solution quality, to generate adaptive mesh)
  - solution error, residual error
  - local, global

• Norms
  - maximum, mean square, energy

• Definitions
  - true low order error $e^{TL} = \|u^T - u^L\|$  
  - true higher order error $e^{TH} = \|u^T - u^H\|$  
  - estimated error $e^{HL} = \|u^H - u^L\| \approx e^{TL}$

• Problem: how to generate a reference solution $u^H$? 
  
  Multipoint MFDM
A posteriori error MFDM analysis – local error estimation

**True** solution
\[ L u = f , \quad \rightarrow u^T \]

**Lower** order solution (standard MFDM)
\[ L u \approx L u = f , \quad L u = f \quad \rightarrow u^L \]

**Higher** order solution (Multipoint MFDM)
\[ L u \approx L u = M f , \quad L u = M f \quad \rightarrow u^H \]

**Local solution error**
\[ e^{TL} = u^T - u^L \quad \rightarrow \quad e^{HL} = u^H - u^L \approx e^{TL} \]
\[ e^{TH} = u^T - u^H \]

**Higher order (Multipoint) estimation** of the local solution error

Various use of **HO approximations**

**Local residual error**
\[ r = L u - f \quad \rightarrow \quad r^L = L u^L - f \]
\[ r^H = L u^H - f \]

**Standard – low order estimation** of the local residuum

**Improved – higher** order estimation of the local residuum
A posteriori error MFDM analysis – global error estimation

Global solution error \( \eta = \| e \| \)

- Error norms \( \| \cdot \| \) used:
  \[
  \| e \|_E = \sqrt{\frac{1}{\Omega} \int_{\Omega} b(e, e) \, d\Omega} \quad \text{energy norm}
  \]
  \[
  \| e \|_2 = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (e_i)^2} \quad \text{mean square norm (discrete } l^2 \text{)}
  \]

Effectivity index of estimator \( I_{eff} = 1 + \frac{\eta - \| e^{TL} \|}{\| e^{TL} \|} \)

Hierarchic estimators
- H-type \( e = u_h^L - u_{h/2}^L \)
- P-type \( e = u_p^L - u_{p+1}^L \)
- MHO-type \( e = u_p^L - u_{p+s}^H \)

Smoothing estimators
- ZZ-type \( e = u_{\text{rough}}^L - u_{\text{smooth}}^L \)

Residual estimators
- Explicit \( e = \sqrt{h^2 \| r \|^2 + h / 2 \| J \|^2} \)
- Implicit \( e \rightarrow b(e, e) = r \)
Error analysis

- Error indicators for irregular meshes

Test 2. Solution convergence based on the series of adaptive meshes using the both types of the error indicators
Benchmark 2D problems

Two-dimensional Poisson’s b.v. problem

$$\nabla^2 u = f(x, y) \quad \forall \quad \Omega$$

with the Dirichlet b.c. \(0 \leq x \leq 1, \quad 0 \leq y \leq 1\)

Test 3

\[u(x, y) = \sin(x + y)\]

Test 4

\[u(x, y) = -x^3 - y^3 + \exp\left(-100(x - 0.5)^2 - 100(y - 0.5)^2\right)\]
Convergence for multipoint & standard MFDM

Solution convergence, results of the general multipoint method and the standard MFDM
Multipoint error analysis

- Hierarchic ($p$-type)

Solution convergence based on the exact mean solution errors for 1D and 2D tests and various approximation orders.
Multipoint error analysis

- Hierarchic ($p$-type)

Test 4. The exact solution error of (i) basic MFDM; (ii) Multipoint MFDM

(i) $\text{max} = 3.0 \times 10^{-2}$

(ii) $\text{max} = 1.75 \times 10^{-4}$
Multipoint error analysis – irregular meshes

Voronoi polygons for random irregular mesh (up to 33% irregularity 289 nodes)

\[ \frac{\Delta x}{h} \leq \frac{1}{3} = 33\% \]

Test 3. Convergence of the average of 20 random irregular meshes and regular ones, all using 3rd approximation order
Multipoint error analysis

- Hierarchic estimator

\[ I_{\text{eff}} = 1 + \frac{\|e^{HL}\| - \|e^{TL}\|}{\|e^{TL}\|} \]

Comparison of the error estimation \( e^{HL} \) and the exact error \( e^{TL} \) for low order solution. Test 3, general Multipoint approach. Effectivity index \(~1.1\)
Multipoint error analysis

- **Effectivity index**

<table>
<thead>
<tr>
<th>True solution error</th>
<th>Test 3</th>
<th>Test 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Max norm</td>
<td>Euclidean norm</td>
</tr>
<tr>
<td>Higher order true error $e^{TH}$</td>
<td>3.0e-6</td>
<td>1.18e-6</td>
</tr>
<tr>
<td>(version 2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Higher order true error $e^{TH}$</td>
<td>1.22e-6</td>
<td>4.12e-7</td>
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<tr>
<td>(version 3), local formulation</td>
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<td></td>
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<tr>
<td>Higher order true error $e^{TH}$</td>
<td>1.8e-6</td>
<td>8.29e-7</td>
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<tr>
<td>(version 3), global formulation</td>
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<tr>
<td>Higher order true error $e^{TH}$</td>
<td>4.0e-6</td>
<td>2.08e-6</td>
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<tr>
<td>(version 4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower order true error $e^{TL}$</td>
<td>2.54e-4</td>
<td>1.24e-4</td>
</tr>
<tr>
<td>Error estimation $e^{HL}$</td>
<td>2.5e-4 – 2.54e-4</td>
<td>1.22e-4 – 1.24e-4</td>
</tr>
<tr>
<td>Effectivity index $I_{eff}$</td>
<td>1 – – 1.02</td>
<td>1 – – 1.02</td>
</tr>
</tbody>
</table>

Error estimation for various Multipoint MFDM versions. Regular 249 nodes mesh.
Multipoint error analysis – prismatic bar twisting

Φ – Prandtl stress function, $G\theta = 1$ – torsional stiffness, $\Omega$ – domain of the bar cross-section.

\[
\begin{align*}
\nabla^2 \Phi &= -2G\theta, & \text{in } & \Omega \\
\Phi &= 0, & \text{on } & \partial\Omega \\
\end{align*}
\]

The total shear stress

\[
\tau = |\text{grad } \Phi| = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}
\]

shear stresses are

\[
\tau_{zx} = \frac{\partial \Phi}{\partial y}, \quad \tau_{zy} = -\frac{\partial \Phi}{\partial x}
\]

- **Railroad rail cross-section**
- **Benchmark test – square cross-section**
Prismatic bar (rail-shape) twisting

Irregular mesh, Prandtl stress function and total shear stress in railroad rail shape bar
Prismatic bar twisting.
True solution error for Prandtl function

Multipoint solution error:
a) local formulation and 2\textsuperscript{nd} app.order,
b) local formulation and 3\textsuperscript{rd} app.order,
c) global-local formulation (MLPG5) and 3\textsuperscript{rd} app.order

\[ e^{TL} = u^T - u^L \]

Exact lower order error

\[ e^{TH} = u^T - u^H \]

Exact higher order error
Estimated 2\textsuperscript{nd} approx. order solution error

\begin{align*}
&\text{Multipoint solution (local formulation) error:} \\
&a) \text{exact error of the 2}\textsuperscript{nd} \text{app.order} \quad b) \text{estimated error by 3}\textsuperscript{rd} \text{app.order} \\
&c) \text{estimated error by the MLPG5 3}\textsuperscript{rd} \text{app.order}
\end{align*}

\[ e^{HL} = u^H - u^L \approx e^{TL} \]

Estimated error
Estimated 3\textsuperscript{rd} approx. order solution error

\[ e^{HH} = u^{H(p1)} - u^{H(p2)} \approx e^{TH} \]

**Multipoint solution error:**

a) exact solution error for the local formulation and 3\textsuperscript{rd} app.order

b) solution error for the local formulation and 3\textsuperscript{rd} app.order estimated by the MLPG5
Multipoint error analysis

- **Residual error distribution**
  
  \[ r^L = Lu^L - f \quad r^H = Lu^H - f \]

Residual error distribution using 1, 9, and 19 points between nodes. Test 2.
A posteriori error analysis

- Residual error

Residual error distribution using 19 points between nodes.
The influence of the MWLS weighting factor $g$
Smaller $g$ provides smaller errors

$$k w_{ij}^2 = \left( \frac{\rho_{ij}^2 + \left( \frac{g^4}{\rho_{ij}^2 + g^2} \right)}{\rho_{ij}^2 + g^2} \right)^{-p+k-1}$$
Multipoint error analysis

- **Smoothing**

Zienkiewicz-Zhu type error estimator

\[ e^{TH} = u^T - u^H \quad e^{LH} = u^L - u^H \approx e^{TL} \]

Specific case

\[ u^{(k)} = C \, u + \alpha \, f \]

General case

\[ u^{(k)} = A \cdot u \]
Final remarks

- Presented was a posteriori error analysis based on the Multipoint Meshless Finite Difference Method (MMFDM).
- The MMFDM may provide high quality reference solutions for a posteriori error estimation.
- The following solution combinations may be done:
  - HO multipoint one vs. lower order standard (MFDM or FEM) solution $u^H - u^L$
  - Two different HO multipoint solutions, e.g. $p+1$ and $p+2$, presenting various use of higher order approximations
- Various types of a posteriori error estimation, including hierarchical, residual and Zienkiewicz-Zhu approaches, may be carried out when supported by the Multipoint MFD method, and applied to both irregular and regular meshes.
- Numerous tests carried out present encouraging, high quality results of such error analysis. However, results of the residual error analysis carried out between nodes show strong dependence of the error value on the choice of the used MWLS smoothing weight factor value $g$.
- Further development and testing of this approach seems to be justified.
Thank you very much for your attention!