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Part I

STRUCTURAL MODELS

Part I of this book begins from the chapter that deals with the understanding of the concept of "structure" in mathematics and the social sciences. Although the worlds studied by mathematics and empirical sciences seem to be worlds apart, one can bridge the ontological gap between them by building *formalized empirical theories* based on *representing* empirical *systems* by *sets* endowed with *structures* of various types. When such a representation is found, structural properties – primarily defined for *mathematical objects* of a given *category* as properties preserved by *isomorphisms* – can be ascribed to empirical systems via their mathematical models. For example, a *social* group can be represented by a *directed graph* G=(N,R) where N, or the set of points of G, stands for the set of group members, and R is a binary relation in N, obtained, for instance, by asking each member of N to name his or her "friends" in N.

What are the benefits of such a representation? Once the graph model ignores the *content* of the modeled empirical relation, the substantive nature of dyadic friendship ties remains unexplicated. However, by representing a "social relationship" as a "relation" – in the mathematical meaning of the term – we gain a *formal language*, which enables us to depict the group structure more rigorously and to state hypotheses about it with the use of *structural variables*. The knowledge of *theorems* about graphs is no less important, as it sheds light on some empirical findings concerning social ties. Consider a hypothesis which states that men have on average more sexual partners than women. Such a claim found support (see Faust 1993) in a survey in which a representative sample of the members of a closed community was interviewed about their heterosexual contacts with other members of that community. However, in any population in which potentially sexually active men and women are equal in number, the *mean* number of female sexual partners of a man equals the *mean* number of male sexual partners of a woman. This statement, for its informal wording, might seem an *empirical* hypothesis. It is, in fact, a mathematical theorem. Anyone who knows relevant terms (degree of a point, bipartite graph) will be able to correctly explain the inequality of the two sample means by pointing to the fact that people too often lie when asked about the number of their sexual partners.

Since the appearance of the interdisciplinary perspective known as *Social Network Analysis* (Wasserman and Faust 1994) the number of researchers in the social and behavioral sciences who are familiar with graph theory language, algorithms and theorems has been systematically growing.

> “One attractive feature of digraph theory for the nonmathematician is its relatively self-contained nature. The mathematical training acquired by most social scientists is sufficient for understanding the contents of this book, although some informal knowledge of logic, a facility in abstract thinking, and that mysterious quality known as ‘mathematical maturity’ will make the task easier.”

Let me redirect these remarks, quoted from the preface to Harary, Norman, and Cartwright's *Structural Models* (1965), to all who start reading my book, another one of the kind. Like the cited classical work, it deals mainly with *graph-theoretic* structural models. A directed graph, defined as a *relational system* (N,R) with *one* relation, is an abstract mathematical object. It can be drawn – hence the name “graph” – yet it should not be identified with none of its many *geometric representations*, or *sociograms*, to use a more familiar term. Let us remark that a visual display of a digraph can also be construed as a structured mathematical object, yet the fact that some structural properties can be “seen” is not in itself an argument for the superiority of visible (“graphical”) over invisible structures.

Chapter 1 owes its length to introductory reflections on the properties of scientific knowledge and my involvement with “foundational” problems of sociology. My principal aim, however, is to show you in this chapter the gist of “structural approach” in mathematics and examine its implications for “structural modeling” in the social sciences. In modern mathematics, “structure” is a generic term that encompasses in its scope many particular “species of structure” such as binary relations, algebraic operations, topological structures, and many others. A metatheoretical analysis of mathematical and sociological “structuralism” will help you see graphs in a broader conceptual framework in which “relation” is no longer a synonym of “structure” but means a special kind of structure.

Chapter 2 begins from a formal analysis of actions and transactions – the most *elementary* events that occur both in *dyadic* exchange systems (called “exchange relations” by Emerson) and larger systems where at least one actor has more than one potential transaction partner. Following a discussion of few formal approaches to dyadic interaction with focus on the Nash bargaining model, we present the basic terminology of graph theory and few theorems to be used in Part II. The aim of the last section of Chapter 2 is to define a *network exchange system* (*exchange network*, for short) as a complex mathematical object obtained by endowing a *transaction opportunity graph* (a graph whose lines specify who can exchange with whom) with a *network* structure (an assignment of *profit pools* to graph lines) and *exchange regime* – a structure that makes bilateral transaction opportunities interdependent. A transaction between two actors will be construed as a division of the profit pool assigned to the line connecting the actors' positions.

Chapter 1

Structural Mathematical Sociology

1.1. Basics of the methodology of the basic sciences

1.1.1. Chapter 1 of this book opens with a lengthy *metatheoretical* overture whose themes come from my earlier papers (Sozański 1995b, 1998, 2002; in Polish). To explain what is meant by *structural mathematical sociology*, I need to discuss first how mathematics, which is a *formal* science, is related to *basic empirical* sciences, in particular, to *exact* sciences. Every *basic science,* no matter formal or empirical, natural or social, "is oriented to the production and evaluation of knowledge claims" where the term *knowledge claim* is referred to any statement which "can be accepted or rejected on the basis of some criterion of truth." (Cohen 1989: 52–53).

The methodology of the basic sciences deals with epistemic criteria for evaluating solutions to *scientific problems*. In every science, the range of problems considered tractable is determined by one or more paradigms, a *paradigm* being defined roughly as a set of guidelines, accepted by the academic community, on *what* and *how* can be studied in a given scientific discipline or subdiscipline.

What are distinguishing features of *science* as a special kind of *knowledge*? To put in another way, what ideals or standards guide the production of this kind of knowledge? The answer is that scientific knowledge is expected to be:

- intersubjectively communicable;
- methodically produced and validated;
- systematized;
- consistent;
- intersubjectively provable or testable;
- as certain as possible;
- rich in information;
- universal;
- general;
- precise and accurate;
- parsimonious and simple;
- abstract;
- conditional;
- cumulative.

Most of these characteristics were also included by Markovsky (1997b) in his list of criteria for evaluating scientific *theories*. The discussion given below will be a bit more technical than his, as I am going to combine metatheoretical considerations with a presentation of few elementary mathematical terms to be used later in this book.

1.1.2. *Intersubjective communicability*, placed first on the list, is being achieved in each discipline through codifying its language. That is to say, there must be established certain clear, objective, workable criteria upon which meaningful statements can be distinguished from those to be considered meaningless. Codification of *scientific discourse* inevitably leads to supplanting natural language by artificial *formal languages* where complex expressions are built from simpler ones by applying to them certain explicitly stated rules so that meaningful statements are easily recognized from their syntactic structure. Formalization of the *syntax* (relations within a system of signs) is a necessary step preceding the codification of two other aspects (distinguished by Morris in his *Foundations of the Theory of Signs*, 1938) of any language, *semantics* (relations between linguistic expressions and the objects in the "world" to which they refer) and *pragmatics* (relations between a language and its users).

1.1.3. Scientific knowledge has to be produced *methodically*, even if it ultimately grows out of unplanned discoveries of new facts or new conceptual representations of known facts. *Methods* are prescriptions on how to perform various activities at all stages of the research process, first of all, at its last and most important stage when knowledge claims are validated upon "some criteria of truth." In the *formal sciences*, a knowledge claim is accepted if and only if it can be deduced from already accepted claims by means of the logical *rules of inference*. The *deductive method* is also used in the *empirical sciences* along with *empirical testing* (in particular, *experimental method*). This second way of validating knowledge claims is peculiar to these sciences. By saying that scientific knowledge must be produced methodically, we also mean that *evidence* needed to test a hypothesis must be collected with the use of intersubjectively controllable *data generation procedures*.

1.1.4. Science differs from the common sense knowledge, in particular, with the degree of *systematization*, which pertains both to terms and propositions, two basic components of any knowledge. *Terms* are names of things, properties, relations, functions, and other constructs studied in a given field. *Propositions* (sentences), as formed with the use of terms, constitute the higher level of the language. What is even more important, they are conceived of as statements which can be *true* or *false* in a given *domain* in which the terms occurring in them are *semantically interpreted*.

Collections of terms and propositions should be structured so as to form terminologies and theories. To develop a *terminology*, one has to point out *primitive terms*, or those terms that must remain undefined so that they could serve as a basis for defining other terms. To narrow down the range of admissible interpretations of primitive terms, one must accept certain *meaning postulates*, or propositions in which these terms jointly occur.

To explain and illustrate metatheoretical concepts, we construct a simple formal language suitable for the study of sex and kinship. Let E and P be two primitive *constant relational terms*. Elementary *propositional formulas* or *conditions* containing these terms will have the form xEy, xPy to be read respectively, as "x is of the same sex as y," "x is a parent of y." Letters x, y which appear in these expressions are called *individual logical variables*. If a and b *denote* two concrete elements of a fixed domain, then the statement aEb ("a is of the same sex as b") can be true or false. The logical value of the statement *xEy* is not determined since *x* and *y represent* unspecified elements of that domain.

To find out if the proposition *aEb* is true or false in a domain whose elements are human beings, one must identify two persons named *a* and *b* and check if they are of the same sex. To verify that the proposition $aEb \vee \neg(aEb)$ ("*a* is of the same sex as *b* or *a* is not of the same sex as *b*") is true, one does not need to examine *a* or *b* nor even interpret the language in any domain. The proposition in question is a *tautology*, which means, by definition, that it remains true in *every* domain solely by virtue of its syntactic form, given in this case as $p \vee \neg p$ where *p* stands for any proposition and symbols $\vee$ and $\neg$ denote *disjunction* and *negation*. The other familiar *logical connectives*, used as *nonspecific terms* in any scientific discipline, are *conjunction* ($\wedge$), *implication* ($\Rightarrow$), and *logical equivalence* ($\Leftrightarrow$). Since too formal style often makes it difficult to grasp the logical meaning of a statement, I will almost always use common English words instead of special symbols (*p or q*, *not p*, *p and q*, *if p then q*, *p if and only if q*, *p iff q*, for short).

The *logical value* (truth or falsity, symbolically, 1 or 0) of a compound proposition formed with the use of logical connectives is uniquely determined by the logical values of its components and appropriate *truth-functions*. The truth-functions that correspond to basic logical connectives are defined by the formulas: $\neg 1=0$, $\neg 0=1$, $1\vee 1=1\vee 0=0\vee 1=1$, $1\wedge 1=1$, $1\wedge 0=0\wedge 1=0\wedge 0=0$, $1\Rightarrow 1=0\Rightarrow 1=0\Rightarrow 0=1$, $1\Rightarrow 0=0$, $1\Leftrightarrow 1=0\Leftrightarrow 0=1$, $1\Leftrightarrow 0=0\Leftrightarrow 1=0$.

A propositional formula containing logical variables can be converted into a proposition by *replacing* some *free* variables with *constants* and/or *binding* other free variables with the *universal* or *existential quantifier*, as shown in the following examples: *aPb* ("*a* is a parent of *b*"), $\exists x(xPa)$ ("there exists an *x* such that *x* is a parent of *a*"), $\forall x\exists y(yPx)$ ("every person has a parent"), $\exists z\forall x(zPx)$ ("there exists a person who is a parent of every person").

Ludwig Wittgenstein said that (*Tractatus logico-philosophicus*, 4.116) "What can be said at all can be said clearly." The language of science is being made clear, first of all, through explicit and codified use of logical variables and quantifiers. What can be said clearly about sex and kinship? Let us begin from stating the meaning postulates which say that *E* is an *equivalence relation* (*E* is reflexive, symmetric and transitive) and there are at most two sex categories.

P1: $\forall x(xEx)$ (*reflexivity*)

P2: $\forall x\forall y(xEy \Rightarrow yEx)$ (*symmetry*)

P3: $\forall x\forall y\forall z(xEy \wedge yEz \Rightarrow yEz)$ (*transitivity*)

P4: $\forall x\forall y\forall z(\neg(xEy) \wedge \neg(xEz) \Rightarrow yEz)$

To characterize the parenthood relation, we state the following postulate of *strict antisymmetry*

P5: $\forall x\forall y(xPy \Rightarrow \neg(yPx))$

which reads "for all *x* and *y*, if *x* is a parent of *y*, then *y* is not a parent of *x*." Given *P* as a primitive term, one can define a number of kinship relations, in particular, the childhood (P^{-1}) and grandparenthood (P^2) relations.

D1: $xP^{-1}y \Leftrightarrow(\text{df})\ yPx$

D2: $xP^{2}y \Leftrightarrow(\text{df})\ \exists z(xPz \wedge zPy)$

Definitions D1 and D2 fall under two common patterns of constructing new relations from those already available in a given language. P^{-1} is the *inverse* of P and P^2, which is short for PP, is the *composition* of P with P.

Note that definitions can't be true or false since they are not propositions. They are rules that allow us to replace the defined expression (*definiendum*) with the defining expression (*definiens*) in any place where the latter is found. To any rule of the kind there corresponds a proposition (e.g., $\forall x\forall y(xP^{-1}y\Leftrightarrow yPx)$ is obtained from D1) which is true under any interpretation given to P, being therefore a tautology in any language that contains the respective definitional convention.

If the meaning of sex and kinship were to depend solely on postulates P1–P5, the two relational concepts would be bound together only through their common domain of reference. To establish a more “intimate” relationship between the two primitive terms, we propose the following additional postulate

P6: $\forall x\forall y\forall z(xPz \wedge yPz \wedge x\neq y \Rightarrow \neg(xEy))$

which reads informally: If a person (z) has two different parents (x,y), then they can't be of the same sex. Notice that postulate P6 contains a statement $x\neq y$, short for $\neg(x=y)$. It is usually tacitly assumed of any logical language to contain *identity* as a *nonspecific* relational term satisfying the meaning postulates of an equivalence relation (i.e. $\forall x(x=x)$; $\forall x\forall y((x=y)\Rightarrow(y=x))$; $\forall x\forall y\forall z((x=y)\wedge(y=z)\Rightarrow(x=z))$. Unlike specific relational terms, identity is always interpreted in a standard way as the binary relation that holds only between identical elements.

While the terms which appear in the meaning postulates provide the foundation for building up multi-tiered terminology, the postulates themselves play the role of assumptions from which more propositions can be deduced. To give an example, consider the following theorem asserting that every person has no more than two parents.

C: $\forall x\forall y\forall z\forall u(xPu \wedge yPu \wedge zPu \Rightarrow x=y \vee x=z \vee y=z)$

Proof. Suppose that it is not the case that C. Then, there exist a,b,c,d such that aPd, bPd, cPd, and $a\neq b$, $a\neq c$, $b\neq c$. Since aPd, bPd, and $a\neq b$, we conclude that $\neg(aEb)$ by P6. Similarly, we get $\neg(aEc)$ and $\neg(bEc)$. Now we use P4 to derive bEc from $\neg(aEb)$ and $\neg(aEc)$. Thus, we arrive at a contradiction ($\neg(bEc)$, bEc), which completes the proof.

The above proof of C meets the standards of precision required in mathematics. At the cost of length, it could be given an even more formalized shape so as to satisfy the conditions required of a proof in logic. While every formal language has *rules of formation* that determine the set of all well-formed expressions, *logical languages* are also endowed with *rules of inference*, or the rules that are used to derive or infer some propositions from other propositions. To give a familiar example, consider the rule known as *modus ponens*. It can be stated in the *metalanguage* as $p\Rightarrow q,\ p \vdash q$ where p and q denote any two propositions and the symbol $\vdash$ is to inform that $p\Rightarrow q$ and p are input to the rule. Proposition q, or the rule's output called the *conclusion*, is said to *follow from* propositions $p \Rightarrow q$ and p, or the *premises* of a reasoning formally described by means of a given rule.

Although inference is a syntactic concept, any valid inference rule has the following semantic property: If the premises are true in a domain, the conclusion is true as well in that domain. The *modus ponens* rule satisfies this fundamental requirement because the formula $((p \Rightarrow q) \wedge p) \Rightarrow q$ is a tautology of the *propositional calculus*, that is, the logical value of the formula equals 1 for any pair of values assigned to p and q. Finally, let us remark that the rules of inference are related to the *pragmatic* concept of *proposition acceptance* via the following principle: Anyone who accepts the premises must also accept the conclusion derived from them.

Let p be a proposition and Z be a fixed set of propositions stated in a formal language of the kind we have been discussing. A *proof* (the term *demonstration* is often used synonymously) *of p from Z* is defined as a finite sequence of propositions $p_1,\ldots,p_n$ such that $p_n=p$ and any p_i meets at least one of three conditions: (1) p_i is in Z; (2) p_i is a tautology; (3) p_i is derived from some propositions preceding p_i in the sequence by means of an inference rule. The elements of Z are called *assumptions*.

To prove a hypothesis (p) under a given set of *assumptions* (Z) is often a difficult task requiring special skills and creative thinking. To make this task easier, logic has worked out some standard tools, in particular, the so called "deduction theorem" and the method known as "indirect proof" or *reductio ad absurdum*. Let $Cn(Z)$ denote the set of *consequences* of Z, or the set of all propositions having proofs from Z. The *deduction theorem* asserts that $p \Rightarrow q \in Cn(Z)$ if and only if $q \in Cn(Z \cup \{p\})$. In other words, if one has to prove an implication $p \Rightarrow q$, one can derive q from the set of assumptions extended by p. We assume that the symbols $\in$ (being an *element* of a *set*) and $\cup$ (*union* of sets) which have just been used to state the deduction theorem are known to the reader along with few other basic terms of set theory.

When it's not clear how to derive p from Z, one can try to deduce two contradictory propositions from the assumptions and the negation of p. Formally, a proof of p by *reductio ad absurdum* is completed as soon as one finds a q such that $q \in Cn(Z \cup \{\neg p\})$ and $\neg q \in Cn(Z \cup \{\neg p\})$. The demonstration given above is an example of an indirect proof. To prove C, we needed only postulates P4 and P6 as assumptions. Thus, C$\in Cn(\{$P4,P6$\})$. If an indirect proof of p is available, then there also exists an ordinary proof of p from Z.

> To construct the direct proof of p, notice that the deduction theorem implies that $\neg p \Rightarrow q \in Cn(Z)$ and $\neg p \Rightarrow \neg q \in Cn(Z)$. Let us join the proofs of $\neg p \Rightarrow q$ and $\neg p \Rightarrow \neg q$ from Z into one sequence of propositions and append at the end the tautology $(r \Rightarrow s) \Rightarrow ((r \Rightarrow t) \Rightarrow (r \Rightarrow (s \wedge t)))$ in which r, s, and t have been replaced, respectively, with $\neg p$, q, and $\neg q$. By applying twice the *modus ponens* rule we derive the proposition $\neg p \Rightarrow q \wedge \neg q$. We append in turn the tautology $(\neg p \Rightarrow q \wedge \neg q) \Rightarrow p$, and apply the same rule once again to obtain p, which completes the proof of p from Z.

Let us go back to explaining the systematization of scientific knowledge. In a weaker sense, it means grouping knowledge claims into blocks united each by the same subject matter, things, certain properties or relationships studied in a given discipline or subdiscipline. Such a *substantive* systematization is usually a prelude to *deductive* systematization, or constructing theories. A collection T of propositions concerning a fixed domain and containing the same set of terms is called a *theory* if the language in which elements of T are stated is endowed with inference rules and $Cn(T)\subset T$. That is, propositions which can be deduced from some propositions in T are in T as well. Then $T=Cn(T)$ because $T\subset Cn(T)$ (any p in T has a one-line proof made up of p). Clearly, any T such that $T\neq Cn(T)$ can be extended to a theory $T'=Cn(T)$ ($T'=Cn(T')$, that is, $Cn(Cn(T))=Cn(T)$). Therefore, the deductive systematization does matter in science inasmuch as it takes the form of *axiomatization*, which amounts to representing a given theory T as the set $Cn(T_0)$ of consequences of a smaller, if possible finite set T_0 of propositions called the *axioms* of T. Mathematical theories are usually constructed by choosing in advance few propositions to play the role of axioms and to serve at the same time as meaning postulates for primitive terms. For example, a formal theory of sex and kinship is obtained as $Cn(\{\text{P1},\dots,\text{P6}\})$.

1.1.5. Contradictory hypotheses may coexist in science, yet among the propositions accepted in a given discipline there should never be two sentences such that one of them is the negation of the other. *Consistency*, defined by this requirement, is the most fundamental condition that any jointly accepted collection of knowledge claims must satisfy. In particular, every scientific theory should be consistent. If a theory T is not consistent, then it contains p and $\neg p$, for some proposition p, and hence it contains $p\wedge\neg p$ as well. Like every theory, T includes all tautologies, in particular, $(p\wedge\neg p)\Rightarrow q$, which implies that every q is in T. Thus, any inconsistent theory coincides with the set of all propositions which can be stated in a given formal language. Such a theory has no descriptive value because there is no possible world in which both p and $\neg p$ are true. Any theory whose all propositions are true in every possible world is substantively useless as well because tautologies do not convey in themselves any information on any world whatsoever. Their role in every formal language is only to enable logical inference.

In order to prove that an axiomatic theory is not tautological, in other words, that it is not part of the *logical calculus*, one needs to show that at least one of its axioms is false under some semantic interpretation of the theory's language. The formal theory of sex and kinship is not tautological because one can imagine a society where parents may be of the same sex. Unfortunately, one need not resort to imagination as there already exist such societies.

Similarly, to make sure that an axiomatic theory is consistent, it suffices to construct its *model*, which amounts to pointing out a domain and an interpretation of the theory's primitive terms under which all axioms are true. To demonstrate the consistency of our illustrative theory, assume that variables $x,y,z,\dots$ run over the set $\{t,m,d\}$ made up of three distinct persons, say, me (t), my wife (m), and our daughter (d). Clearly, axioms P1,…,P6 are true if P and E are the names of two sets of ordered pairs: $\{(t,d),(m,d)\}$ and $\{(t,t),(m,m),(d,d),(m,d),(d,m)\}$, respectively. In general, every model of a theory with two primitive relational terms is a *relational system* of the form (X,P,E) where X is a nonempty set and P and E denote two binary relations in X. Any relation in a set X is also a set – a set constructed from elements of X. The construction is based on the theory that deals with *sets* and their *elements*, the two notions being tied with each other by the *axiom of extension*: Two sets are identical if and only if they have the same elements, symbolically, $X=Y \Leftrightarrow \forall x(x\in X \Leftrightarrow x\in Y)$

While finite sets like $\{t,m,d\}$ can be defined by listing its elements "by name," any set whose elements are not known in advance must be construed as a semantic counterpart of some quality (predicate). Since the claim that every intuitively meaningful property can be represented by a set must be rejected as leading to contradiction, some restrictions on defining sets are imposed within traditional (Zermelo-Fraenkel) axiomatic set theory. The ontology of mathematics can also be developed so as to admit of the existence of macro-sets, called *classes,* which consist of sets and behave like sets in many respects, but may not be identified with sets.

Fortunately, to construct models of simple formal theories, one doesn't have to be familiar with the foundations of mathematics. What every mathematician must know are just few uncontroversial axioms which guarantee the existence of the following sets: the empty set $\varnothing$, the (unordered) *pair* $\{a,b\}$ made up of any objects a and b, the *union* of a *family* of sets (the name "family" is usually referred to a *set* whose elements are known to be complex entities, say, they are sets), the *power set* $\mathcal{P}(X)$ (the family of all subsets of a set X), and the set $\{x\in X: v(x)\}$ of those elements of a set X which satisfy a condition v stated in any logical language with individual variables representing elements of X.

These elementary axioms entail the existence of the set $X\times Y$ of all *ordered pairs* of the form (x,y) where x is an element of a set X and y is an element of a set Y. $X\times Y$ is termed the *Cartesian product* of X and Y; its subsets are called *relations between* X and Y. *Binary relations in* X are subsets of $X\times X$, in other words, they are elements of $\mathcal{P}(X\times X)$. We say that a relation $F\subset X\times Y$ is a *mapping of* X *into* Y, symbolically, $F:X\to Y$, if, for any $x\in X$, there exists a $y\in Y$ such that $(x,y)\in F$, and, for any $y'\in Y$, $(x,y')\in F$ implies that $y'=y$. The element of Y that is assigned to x by F is written $F(x)$ as being uniquely determined by x and F; it is called the *value* of F for x or the *image* of x under F. For any $A\subset X$, the *image* of A under F is defined as the set $F(A)=\{y\in Y: y=F(x)$, for some $x\in X\}$. If $F(X)=Y$, that is, every element of Y is the image of an element of X, we say that F is a *surjective* mapping or a mapping of X *onto* Y. We say that F is *injective* or *one-to-one* (1–1), if $x\neq x'$ implies that $F(x)\neq F(x')$, for any $x,x'\in X$. If F is both surjective and injective, it is termed *bijective*. Then, the inverse relation F^{-1} ($(y,x)\in F^{-1}$ if and only if $(x,y)\in F$) is a 1–1 mapping of Y onto X.

For any relation $F\subset X\times Y$ and any relation $G\subset Y\times Z$ the *composition* $G\circ F$ (noted also GF) of G with F is defined as the subset of $X\times Z$ made up of ordered pairs (x,z) such that, for some $y\in Y$, $(x,y)\in F$ and $(y,z)\in G$. If F and G are mappings, then the composition of G with F is a mapping of X into Z: the value of $G\circ F$ for any $x\in X$ equals $G(F(x))$. If F is a 1–1 mapping of X onto Y, then $F^{-1}\circ F=I_X$ where I_X is the *identity mapping* of X onto X: $I_X(x)=x$, for any $x\in X$.

No science can do without using set-theoretical notions to represent what it studies no matter whether it deals with tangible or abstract entities. Wójcicki (1979: 137) is right to say that "all sentences, including those of the conversational language, have some mathematical structure (they can be interpreted in a set-theoretical manner) and consequently, all theories can be viewed as mathematical, although perhaps completely trivial ones."

Some philosophers believe that mathematics provides no nontautological knowledge of any reality lying outside the symbolic language because mathematical theorems are merely pieces of formal text derived from other pieces according to the rules of a logical "language game." Contrary to this claim, most mathematicians have always developed their theories with the aim to discover nontrivial facts about numerical or other more abstract domains.

1.1.6. The set $\mathbb{N}=\{0,1,\ldots\}$ of *natural numbers* is certainly the most familiar mathematical domain. Counting things is the simplest use of mathematics both in science and everyday life. *Arithmetic,* or the axiomatic theory of natural numbers, owes its *methodological* importance for all mathematics to the *axiom of mathematical induction.* To prove that all terms of a countable sequence satisfy some condition, one needs to make sure that it holds for the first term and to show that whenever the nth term meets the condition, then so does the $(n+1)$th term.

Natural numbers are also used to endow any logical language with *indexed variables* x_n, $n=0,1,\ldots$ as well as to define terms dependent on n, such as nth *power* of a fixed binary relation. Let $P^0=I$ where I is the identity relation ($xIy \Leftrightarrow$(df) $x=y$). Let P^{i+1} stand for the composition of P with P^i (that is, $xP^{i+1}y \Leftrightarrow$(df) $\exists z(xPz$ *and* $zP^iy)$. Thus, if P^i is defined, then P^{i+1} is defined as well. The axiom of mathematical induction implies that P^n is defined for every $n\geq 0$. For $n\geq 2$, P^n can also be defined as follows:

D3: $xP^ny \Leftrightarrow$(df) $\exists x_1\ldots\exists x_{n-1}\,(xPx_1\ldots x_{n-1}Py)$.

D2, or the definition of the grandparenthood relation, is a special case of D3. The ancestorhood relation will be defined as the *transitive extension* of P.

D4: $xP^{\mathrm{tr}}y \Leftrightarrow$(df) $\exists n\geq 1(xP^ny)$

If arithmetic is to serve as a basis for other formal theories, it should be consistent itself. Unfortunately, some 80 years ago Gödel proved the impossibility to derive the consistency of arithmetic from its axioms alone. To prove that a formal theory is consistent by constructing its model, one must *assume* the consistency of the theory that is used to justify the construction. Regardless of whether arithmetic or the theory of sets is construed as an independent theory or as part of the more fundamental theory of sets, the consistency of the latter theory, or the one which provides means for constructing models of other theories, is no less problematic. Although these discoveries have shaken the confidence in mathematics as a science with particularly firm foundations, I believe with Bourbaki (1968: 13) that "the essential parts of this majestic edifice will never collapse as a result of the sudden appearance of a contradiction; but we cannot pretend that this opinion rests on anything more than experience."

1.1.7. Since "experience" matters not only in the empirical sciences, it would be unreasonable to expect from *any* science to arrive at *absolute* nontautological truths. We require from scientific knowledge to be *intersubjectively provable* or *testable*, but we have to put up with the fact that all proofs and tests are *relative*. In the *formal sciences*, a hypothesis is accepted as a *theorem* if there exists its demonstration based on explicit specific axioms whose consistency is usually justified by invoking a more fundamental theory. The *empirical sciences* use the *deductive method* as well – as a way to derive consequences from already accepted theoretical propositions and as part of testing procedures that are peculiar to these sciences, being their main method of evaluating knowledge claims.

To *test* an empirical theory, one must first identify a number of situations that meet the theory's *scope conditions* and admit of gathering *evidence* indispensable for validating theoretical predictions. The scope conditions (see Cohen 1989: 83; Foschi 1997) determine the range of systems to which the theory applies; they can also specify special system states or some additional circumstances in which theoretically predicted events should occur. Since empirical systems that meet all scope conditions are seldom found in nature, science can't do without constructing fully or partially artificial systems. Created by the researcher, they are easier to study than natural systems, but are no less real than the latter.

For any empirical system satisfying a theory's scope conditions, one must state more or less specific hypotheses concerning its predicted “behavior” and derive them from the theory, supplemented, if necessary, with auxiliary assumptions, for example, those concerning representing theoretical variables by operational ones. To test theory-based predictions and thus the theory itself, one must register actual behavior of the system under study, which basically amounts to recording values of some *variables* (we define this term in the next section). If the system's observed behavior agrees with the predicted behavior within the *margin of error* (acceptable in a given discipline), then the theory is said to have been corroborated by the evidence generated to test it.

If observation of a “natural” course of events can't provide sufficiently rich and unambiguous evidence, one has to create an artificial setting in order to give nature an opportunity to speak in a more extensive or more articulate way. In either case, the researcher must devise a procedure to generate evidence interpretable in the context of his theory, or a procedure for translating the cues emitted by the external world into meaningful data. In an ideal world, such a procedure would be dictated by the theory alone. In the real world, it should be designed so as to minimize “error” or “noise” occurring also in experimental systems, as they are made from the material found in the real world and are never completely protected against the influence of external environment.

Given an adequate research design and reliable measurement techniques, the outcome of a test should depend on whether the theory undergoing verification correctly depicts regularities operating within a well defined category of things or events. An empirical theory must be supported by the evidence in a number of tests to get incorporated into the body of established knowledge in a given discipline. Theoretical propositions which have been accepted and have few other desirable properties (universality and generality being considered most important) are called *laws*. Laws are distinguished from *hypotheses* awaiting acceptance or rejection. Once accepted, an empirical law can be applied outside the setting in which its predictive power has been confirmed. Although our confidence in a law grows with each successful application thereof, certainty that characterizes mathematical knowledge can never be attained in empirical sciences. While a mathematical theorem, once correctly demonstrated, is accepted forever, empirical laws are vulnerable to refutation. However, an empirical law need not be automatically discredited if negative results of further tests raise doubts about its validity. If some observations depart from predictions deduced from a well established theory, the first suspicion is that the theory has been incorrectly applied. Such an explanation is possible because scientific knowledge is and must be *conditional*, that is, any scientific knowledge claim is applicable wherever definite scope conditions are met. Whether these conditions are or are not met is a matter of empirical knowledge.

The core laws of an empirical theory that are protected from hasty falsification are called *principles.* Their epistemic status is the most contentious issue in the philosophy of science. While, for realists, principles render objective *regularities* that are *discovered* by science in nature, for conventionalists, they are *instruments invented* to enable a selective, concise and coherent account of the data.

As Lakatos noticed (1970), an empirical theory does not drop out of the corpus of accepted scientific knowledge because of being simply falsified. Once approved, a theory is abandoned only if it can be replaced by a new theory which accounts for all facts explained by the old theory as well as for some facts that the latter can't explain. It is the strongest meaning of the postulate that scientific knowledge should grow *cumulatively*.

If two theories with the same scope contend for acceptance, it would be pointless to collect more data corroborating each of them separately. An investigation should be designed so as to obtain an empirical basis suitable for testing these theories against each other. Similarly, the evidence gathered for a judicial investigation should allow the court of law to point out one of the previously identified suspects as the most likely perpetrator.

1.1.8. Every *investigation*, scientific or judicial, theoretically or practically oriented, is aimed at reducing cognitive *uncertainty*, first of all, in any situation where hypothetical answers to a question are known, but one is not sure of which of them is true.

The range of questions considered meaningful in an empirical discipline depends on a particular formal representation of a class of phenomena to be studied. The role of specific theories that are formulated within a given conceptual framework is to determine or narrow down the set of plausible answers to these questions. Information provided by the data is needed in turn to reduce uncertainty about which answer to accept. If one were able to assign *probabilities* p_i to mutually exclusive hypotheses $s_1,\ldots,s_n$ ($\sum p_i=1$), the degree of uncertainty in this situation could be estimated by means of Shannon's *entropy function* $H(p_1,\ldots,p_n)= -\sum p_i \log_2 p_i$ which attains the minimum value 0 if and only if $p_j=1$, for some j, that is, maximal certainty is attributed to exactly one hypothesis. If two contradictory statements, s_1 and s_2, are believed to be equally likely ($p_1=p_2={}^1/_2$), the degree of uncertainty equals 1. A reduction of uncertainty from 1 to 0 is referred to as the reception of 1 *bit* of information.

“In a somewhat aphoristic form, science is an information-seeking process” (Szaniawski 1976: 297). In light of formal *information theory*, richness of *information* and *certainty*, two items on our list of the goals pursued by science, turn out to be conceptually intertwined. However, their understanding must remain intuitive until an *intersubjective* practical method for measuring epistemic probability becomes available. In general, the pragmatic aspect of the language of science admits of limited codification, which opens the door for sociological interpretations of methodological rules as norms or conventions approved by academic communities.

1.1.9. We discuss in turn *universality* and *generality*, two highly evaluated qualities that distinguish the *laws* from other accepted scientific propositions. "A universal statement is a statement whose truth is independent of time, space, or historical circumstance" (Cohen 1989: 78). To ascertain whether an empirical theory is universal, one must put it to the test in at least two settings that differ with space-time or sociocultural coordinates. In the social sciences, "the cross-national and cross-cultural replication experiment is the only method of testing a theory for universality" (Szmatka 1997: 95). To carry out such a test for a *sociological* theory, one must build an empirical system satisfying the theory's scope conditions within a different sociocultural environment. This may appear unfeasible, even if the theory is universal, as the conditions in which a sociological *law* is applicable may seldom occur in the social universe. Thus, what universality does mean is that the law predicts the behavior of any *relevant* empirical system regardless of "historical" circumstances, or those qualities of the system which have been recognized as inessential and thus not included in the scope conditions.

According to Cohen (1989: 178), universality and deductive systematization are both required of a collection of conceptually interrelated testable statements in order that it can be called an *empirical theory*. If universality is skipped as a too restrictive condition of theoreticity, it returns as the basis of the traditional distinction between *nomothetic* and *idiographic* (*historical*) sciences, the former being defined as those capable of producing universal theories. The scope conditions of a universal theory do not state when and where to find systems to which the theory applies. Nevertheless, one must show that such systems do exist within the real world because otherwise the theory would not be testable. An empirical theory which does not claim universality, in order to be testable, must also have a definite scope to be specified then by indicating the time, place, nation or culture where the theoretically predicted regularities should occur.

In sociology, empirical "theories of the middle range" compete with "total systems of sociological theory" (Merton 1968) or old and new "theories of the first generation," as Szmatka and Sozański (1994: 225–231) called general conceptual maps of the social world at large as well as proper theories that depict the workings of particular social systems, but suffer from the lack of testing procedures or explicitly stated scope conditions. In the same paper, sociological theories that are free from these deficiencies were divided into two other "generations." The latter word was later replaced by Szmatka and Lovaglia (1996) with "genera" to acknowledge the fact that none of the three kinds of theorizing that co-exist in contemporary sociology is going to supersede others in the foreseeable future.

Sociological theories of the third genus, unlike those of the second genus, are universal and abstract. In relation to *natural sciences*, Toulmin (1953: 44–56) used much similar criteria to contrast "physics" with "natural history." Faithful *description* of things and events in the real world is not the only concern of "natural historians." They want to *explain* facts by means of "general laws" of the form "*all* As are Bs." "But so long as one remains within natural history there is little scope for *explaining* anything: 'Chi-chi is black because Chi-chi is a raven and all ravens are black' is hardly the kind of thing a scientist calls an explanation." (Toulmin 1953: 49).

Lee Freese has drawn a similar distinction between the "generalizing view" and the "instrumental view" of theories and laws.

> "Laws [viewed instrumentally] may be construed as nomothetic statements expressed in universal or statistical form and having high information content, but they are not meant to be generalizations about the world of everyday experience. The regularities they describe exist in a theoretically possible world but not in the actual world. … If theories are construed as describing some idealized state of affairs in a closed system, with laws describing the invariances of the system, then they are devices for calculating changes in the system when other things are equal. Though other things are never equal outside of the closed theoretical system … laws may serve as tools for engineering some change in an open empirical system whose departures from some theoretically true state of affairs can be measured." (Freese 1980: 191–192).

The *instrumental* view of laws, which is peculiar to the theories of the third genus, is apparently incompatible with the realist stance in the philosophy of science. However, a law, which in its abstract form applies directly to a class of "theoretically possible" systems, applies indirectly to relevant real-world systems. Its successful indirect *application* to an "open system" is possible due to universality. "Historical embeddedness," which makes an empirical system "open," is not strong enough to "disable" the law that still provides correct predictions, the more accurate, the closer the system to its theoretical *model*.

1.1.10. Universality should not be confused with *generality* being an independent, important characteristic of a law or theory. The broader the scope of a theory, the greater its generality, regardless of the nature, abstract or historical, of entities dealt with by the theory.

In logic, the term *general statement* is referred to any proposition which states that *all* objects have some property, which is written symbolically as $\forall x(v(x))$, where v is a fixed property of objects represented by a logical variable x. Since, for any individual constant c, the implication $\forall x(v(x)) \rightarrow v(c)$ is a tautology, each individual sentence $v(c)$ predicating that c has property v can be *deduced* from the general proposition attributing this property to *all* elements. The derivation of a *particular* conclusion from the general premise is probably the best known pattern of deductive reasoning ("all men are mortal, therefore, Fidel is mortal"). However, generality is in fact a semantic concept because the phrase "all things" acquires a definite meaning no sooner than with pointing out a set X whose elements (or rather their names) are to be substituted for x. The intended semantic interpretation of a general proposition is marked explicitly by writing $\forall x \in X(v(x))$ instead of $\forall x(v(x))$. Then it becomes clear that generality is a relative property. If $Y \subset X$ and $Y \neq X$, then the statement $\forall x \in X(v(x))$ is said to be *more general* than $\forall y \in Y(v(y))$. If X is a set of n objects labeled $c_1, \dots, c_n$, then $\forall x \in X(v(x))$ is equivalent to the conjunction of n propositions

$v(c_1),\ldots,v(c_n)$. Hence, the universal quantifier is indispensable in the scientific discourse, insofar as the laws are meant to be *strictly general* statements, that is, they should hold true in domains containing infinitely many objects.

General statements are usually arrived at by *induction*. First, there must be known *some* things that do have a given property v. Next, one must specify the boundaries of a set X whose elements are expected to share this property. Lastly, when no c such that *not* $v(c)$ has been found in X, one claims that *all* x in X are v. Consider a formalized empirical theory, obtained from our formal theory of sex and kinship by *interpreting P* as the relation of biological parenthood in a population X of all beings of a fixed species which have ever lived, live or will live. If P5 is replaced with a stronger postulate

P5': $\forall x \forall y(xP^{tr}y \Rightarrow \neg(yPx))$,

(which means that a descendant of any creature cannot be its parent), then the general proposition $\forall x \in X(\exists y \in X(yPx))$ can be true only in an infinite domain. Indeed, if X is a finite set, then one can prove the existence of at least one x in X such that *not* yPx, for all y in X (see Theorem 3.8 in Harary, Norman, and Cartwright 1965: 64). Hence, if one construes a species as a *finite* population of organisms living in the real world, then the claim that "*every* creature of given species has a parent of the same species" can't be accepted, regardless of how hard is to find an organism that would not owe its life to the event in which another organism of the same species had been involved.

Do *infinite* domains exist within the *real* world? If not, then the requirement of strict generality may not be fulfilled by any theory of the second genus, unless its scope conditions are modified so as to cover cases existing virtually in the *possible* world. Such a scope extension is the first step toward constructing a theory of the third genus.

1.2. Exact sciences

1.2.1. Endeavors to generalize a theory as much as possible may result in disregarding other no less important goals of science that are usually easier to achieve under more restrictive scope conditions. Generality really counts only if it goes together with *precision* and *accuracy*, as is the case, for instance, with Newton's laws of motion, which not only apply to a broad class of mechanical systems, but yield specific, quantitative predictions which agree remarkably well with measurement results. "Although a theory may generate predictions that are highly precise, the *accuracy* of those predictions – their correspondence to empirical observations – may vary" (Markovsky 1997b: 19). There exist sociological theories which offer exact predictions of the behavior of some social systems, yet the gap between observed and predicted results is often too wide and contingent on uncontrollable events. Hence, the social sciences on the whole cannot yet be counted among *exact sciences*, or those nomothetic empirical sciences that meet the standards of precision and accuracy to a high degree.

In all empirical sciences, the quest for precision requires the transition from concepts to variables. To transform a *concept* (e.g., that of income) into a *variable*, one must begin from selecting an appropriate *unit of analysis* – if the concept admits more than one option in this matter (in the case of income, the unit may be the person, household, firm, etc.). Next, the *domain* of the variable should be pointed out – as the set of objects varying in the respect considered important by the researcher. Which objects of the chosen type will be included in the domain depends on intended generality of the hypotheses to be stated with the use of the variable. Lastly, certain *values* must be assigned to the elements of the domain so as to reflect the differences among them. Thus, in set-theoretic terms, a variable is a *mapping* of a set of objects into a set of values. Variables – in this meaning – should not be confused with *logical* variables. The latter are symbols (in formalized languages) or common nouns (in natural languages) that enable us to speak of things, points, numbers, or other entities without the necessity to point out concrete elements of appropriate sets.

It is convenient to identify all variables having the same domain X with the mappings of X into the set $\mathbb{R}$ of *real numbers*. The role of mathematics in empirical sciences does by no means reduce to the use of numbers, yet greater precision is usually attained in exact sciences through replacing *qualitative* with *quantitative* statements. When you can recognize whether two things x and y do or do not differ from each other in some respect, you would like to also know *how much* differ the two things that differ, which requires defining a variable V and computing the *difference* $V(x)-V(y)$ of the values of V for x and y, or the *distance* $|V(x)-V(y)|$ between them – if the order of values is unimportant.

Variables are classified according to many criteria. The properties of the set of values a variable may assume are taken into account (*discrete* vs. *continuous* variables) as well as the nature of the objects that form the variable's domain (*individual* vs. *collective* variables; see Lazarsfeld and Menzel 1964) or methods for assigning numbers to objects.

The major benefit of the use of real numbers lies in the possibility to express various relationships between variables with the use of *mathematical structures* (order, algebraic operations, distance, topology) available in $\mathbb{R}$. However, for some variables, numerical values must serve solely as names for certain mutually exclusive attributes. In particular, any *property* of elements of X – equated with a subset A of X – can be represented by a variable V_A such that $V_A(x)=1$ if $x \in A$ and $V_A(x)=0$ if $x \notin X-A$. The variables associated with X and $\varnothing$ are *constants* assuming each only one value, 1 or 0, respectively.

If neither A nor $X-A$ is empty, then the two-element family $\{A,X-A\}$ is a partition of X. The term *partition of X* is referred to any nonempty family $\mathcal{A}$ of nonempty sets such that their union is X and the intersection of any two different sets in the family is empty. Any partition $\mathcal{A}$ of X generates the equivalence relation $R_{\mathcal{A}}$ in X, $xR_{\mathcal{A}}y$ being defined by the condition: $x,y \in A$, for some $A \in \mathcal{A}$, that is, any two elements x and y in X are equivalent if and only if they are members of the same set in $\mathcal{A}$. Conversely, to any equivalence relation R in X there corresponds the partition $\mathcal{A}_R$ of X into equivalence classes with respect to R. The *equivalence class* of an $x \in X$ consists of all elements in X which are equivalent to x, symbolically, $[x]_R=\{x' \in X: xRx'\}$. Since $x \in [x]_R$, equivalence classes are nonempty and their union is X. Symmetry and transitivity of R imply in turn that either $[x]_R=[y]_R$ (if xRy) or $[x]_R\cap[y]_R=\varnothing$ (if *not* xRy). Therefore, the family of equivalence classes is a partition of X.

To illustrate the correspondence between partitions and equivalence relations, consider relation E which has been introduced to formalize being of the same sex. Assume that there exist at least two nonequivalent elements in X, say, a and b. Then, P4 implies that the partition of X generated by E consists of exactly two sets, $[a]_E$ and $[b]_E$. If an element m is chosen to play the role of the "standard male," then aEm or bEm. Suppose that aEm. Then, the sets $[a]_E=[m]_E$ and $[b]_E=X-[m]_E$ can be termed, respectively, the set of "males" and that of "females."

1.2.2. We say that a variable T represents a partition A if, for any $x, y\in X$, x and y are in the same class in A if and only if $T(x)=T(y)$. Thus, to represent the sex partition by a variable, it suffices to assign any two distinct real numbers to the two sex groups. In particular, put $V(x)=1$, for $x\in[a]_E$, and $V(x)=0$, for $x\in[b]_E$. Another representation of the same partition is obtained by putting $V'(x)=0$, for $x\in[a]_E$, and $V(x)=1$, for $x\in[b]_E$.

Let variable T represent a partition A and γ be a 1–1 mapping of $\mathbb{R}$ onto $\mathbb{R}$. Then the composition of γ with T represents A as well. The above two representations of the sex partition are related to each other in such a way because $V'=\alpha\circ V$, where $\alpha(r)=1-r$, for all $r\in\mathbb{R}$.

Let Γ_0 denote the set of *all* 1–1 mappings of $\mathbb{R}$ onto $\mathbb{R}$. Notice that if $\gamma, \gamma'\in\Gamma_0$ then $\gamma\circ\gamma'\in\Gamma_0$ and $\gamma^{-1}\in\Gamma_0$. Let Γ be a nonempty subset of Γ_0 having the same properties, that is, the composition of any two mappings in Γ and the inverse of each of them are in Γ as well. Variables T and T' defined on X are said to represent the same *construct of type* Γ if there exists a γ in Γ such that $T'=\gamma\circ T$. For example, let Γ_1 stand for the subset of Γ_0 made up of *order-preserving* mappings, where γ is said to preserves order if $r\leq r'$ implies $\gamma(r)\leq\gamma(r')$, for any $r, r'\in\mathbb{R}$. Variables V and $V'=\alpha\circ V$, which represent a construct of type Γ_0, may not represent the same construct of type Γ_1 because there is no β in Γ_1 such that $V'=\beta\circ V$.

In set-theoretic terms, constructs of type Γ are equivalence classes generated by the relation of Γ-equivalence that is defined in the set of all variables on X by the condition: T' is Γ-equivalent to T if and only if $T'=\gamma\circ T$, for some $\gamma\in\Gamma$. Constructs of type Γ_0 and Γ_1 are called, respectively, *nominal* and *ordinal*, which terms are also referred to variables representing these constructs.

Since "measuring" nominal constructs amounts to classifying objects and attaching numerical labels to the classes, the term "measurement" (see Suppes and Zinnes 1963) is usually reserved for such ways of assigning numbers to things or events that numerical values preserve a weak order relation that is assumed to exist in X prior to being represented by the relation $\leq$ in $\mathbb{R}$. The term *weak order* will be referred here to any binary relation in X (noted with same symbol as its counterpart in $\mathbb{R}$) that meets the conditions of *transitivity* (for any $x,y,z\in X$, $x\leq y$ and $y\leq z$ implies $x\leq z$) and *completeness* ($x\leq y$ *or* $y\leq x$, for any $x, y\in X$). A variable T is said to *preserve* a weak order $\leq$ in X if, for any $x,y\in X$, $x\leq y$ if and only if $T(x)\leq T(y)$. Then, $x<y$ iff $T(x)<T(y)$ and $x\simeq y$ iff $T(x)=T(y)$, where $x<y$ stands for *not* $y\leq x$ and $x\simeq y$ stands for $x\leq y$ *and* $y\leq x$. The relation < defined so in X, like its counterpart in $\mathbb{R}$, is strictly antisymmetric and transitive, while ≃ is an equivalence relation; the respective equivalence classes can be named the *levels of a measurable construct*.

Suppose that $x_1<x_2<x_3$ so that $T(x_1)<T(x_2)<T(x_3)$ if T preserves the weak order in X. The number $l_T(x_1,x_2)=T(x_2)-T(x_1))$ is referred to as the *length* of *interval* $[x_1,x_2]=\{x\in X: x_1\leq x\leq x_2\}$ or the *distance* between x_1 and x_2. Two intervals, $[x_1,x_2]$ and $[x_2,x_3]$, are of equal length or one of them is longer than the other, yet the result of length comparison always depends on which of many variables representing the same ordinal construct has been picked to compute the interval length. For example, if three levels of competence are assigned numbers 0, 1, 2, then the distances between successive levels are equal to each other. If the same three levels are coded with numbers 0,1,3, then the middle level is closer to the lower than to the upper level. The order of distances will be reversed if another ordinal *scale* is used, say, the one with values 0,2,3.

Scale-invariant comparisons of interval lengths are possible for *interval constructs* defined by admitting as *scale transformations* solely order-preserving *linear mappings* of $\mathbb{R}$ onto $\mathbb{R}$. Let Γ_2 denote the set of these mappings. Any element of Γ_2 is determined by two real numbers a and b, where $a>0$, the value of $\gamma_{a,b}$ for any $r\in\mathbb{R}$ being given by the formula $\gamma_{a,b}(r)=ar+b$. The assumption that $a>0$ implies that $\Gamma_2\subset\Gamma_1$.

Let T and T be two Γ_2-equivalent variables, that is, $T=\gamma_{a,b}\circ T$, for some a and $b>0$. The lengths of an interval $[x,y]$ computed with the use of T and T are related to each other by the formula $l_T(x,y)=al_T(x,y)$, which implies that the ratio $l_T(x_1,x_2)/\ l_T(y_1,y_2)$ of two interval lengths will not change if T is replaced with another scale representing the same construct of type Γ_2. For example, two intervals of length equal to 20 and 10 Celsius degrees will have the lengths of 36 and 18 Fahrenheit degrees (F°=(9/5)C°+32). The greater of two intervals is twice as long as the smaller under both scales that measure the same interval construct. The ratio of the numbers assigned to two temperature levels does not share this property, for instance, the ratio of 20 to 10 changes to 68/50=1.36. The value ratio becomes a scale-invariant quantity if the range of admissible scale transformations is restricted to $\Gamma_3=\{\gamma_{a,b}: b=0\}$. Ratio constructs, or those of type Γ_3, admit of the existence of an *absolute zero*, or the level which receives value 0 under every scale representing the construct. Invariant assignment of *all* numbers characterizes *absolute constructs*, or those of type $\Gamma_4=\{\iota\}$, where ι stands for the identity transformation ($\iota(r)=r$). Each absolute construct is represented by exactly one variable.

Construct types, also called *measurement levels*, are naturally ordered from nominal to absolute according to narrowing range of admissible scale transformations ($\Gamma_0\supset\Gamma_1\supset\Gamma_2\supset\Gamma_3\supset\Gamma_4$). Although new invariant properties add at each level to those inherited from the previous level, the leap from ordinal to interval level turns out most significant, which yields the dichotomous division of constructs types into *qualitative* (types Γ_0 and Γ_1) and *quantitative* (types Γ_2. Γ_3, Γ_4).

These terms as well as the names of construct types are also referred to variables. A real-valued mapping T of a domain X becomes a *variable of type* Γ by virtue of the researcher's decision to consider all variables Γ-equivalent to T as interchangeable representations of some structure in X (operationally defined or theoretically assumed to exist). Interval or higher level of measurement is commonly *presumed* by sociologists for many variables, in particular, those obtained by having people respond to close-ended questions where intensity-ordered answers, ranging, say, from "strongly disagree" to "strongly agree," are assigned values of 1 through 5 or -2, -1, 0, 1, 2. The only rationale behind the stipulation that such variables are quantitative is usually the need to underpin the practice of calculating for them the arithmetic mean and other statistical parameters. This practice, which Torgerson (1958: 21–25) aptly named *measurement by fiat*, contrasts sharply with *fundamental measurement* where the validity of a procedure by means of which an empirical domain is mapped into $\mathbb{R}$ rests on a relevant empirical theory. The choice of a measurement level is then no longer made arbitrarily because the same theory determines how any two acceptable numerical representations of an empirical domain should be related to each other.

The third, indirect kind of measurement, known as *derived measurement*, consists in applying a mathematical operation to two or more variables that are directly measurable in the fundamental way. The validity of such a construction that yields a new variable is also warranted by an appropriate empirical law. For instance, density – a variable characterizing substances – is defined as the ratio of mass to volume in virtue of the law which states that the ratio of these two quantities takes a constant value for every amount of a fixed homogenous substance.

1.2.3. In the empirical sciences, variables serve to formulate theoretical hypotheses and their directly testable consequences. What can be studied for a single variable is only the *distribution* of values the variable assumes in a set of objects. Given two or more variables, one wants to know how their values *co-vary* over the common domain. To construct a theory whose axioms have the form of interrelated "covariance hypotheses" (Blalock 1969), one has to select a set of variables and decide on which of them will play the role of *independent variables* in relation to the remaining variables, called *dependent*, whose values could be predicted with a negligible error whenever the values of the independent variables are known. The independent variables are assumed to vary independently of one another, their values occur in many diverse if not all configurations; in experimental systems, it is the researcher who has to make sure that this condition is met.

Although empirical theories are often constructed so as to discover and formally express "causal" linkages among variables, it is the notion of "dependence" rather than that of "causality" that has gained a more "technical" meaning in the language of science. In the simplest case of two variables V and U, V is said to be *functionally dependent* on U if there exists a mapping F of $\mathbb{R}$ into $\mathbb{R}$ such that $V(x)=F(U(x))$, for any $x\in X$, where X is the domain of U and V.

Variables or quasi-variables (concepts that lend themselves to being transformed into variables) appear in all three 3 genera of sociological theorizing. Theories of the first generation abound in propositions of the form "the greater the *division of labor*, the greater the *organic solidarity*." To convert such a covariance statement to a testable hypothesis, one has to recast the two concepts as regular variables, V (the dependent variable) and U (the independent variable). If a "translation" from the first to the second generation of theories is intended, the common domain of these variables mus be defined as a collection of historical social systems. Having assumed interval measurement (*by fiat*) for the two variables, one could try in turn to express the dependence of V on U by means of a *linear* equation $V=aU+E$ where the error variable E is added to the *predicted dependent* variable aU in order to account for its deviation from the *observed dependent* variable V. To reduce the size of error, one could use more than one variable to predict the values of the dependent variable (with two independent variables the linear dependence could have the form $V=a'U+b'U'+E'$).

Second generation theories are expected to provide systematic account of multidimensional differentiation that is actually observed in natural social settings where regularities usually occur in a blurred form due to complex and casual ties within the multitude of variables operating in every concrete population. When the main sources of variation and specific patterns of dependence cannot be identified prior to data collection – for the lack of a "theoretical model" – one may try to construct a "methodological model" (Skvoretz and Fararo 1998), or extract regularities directly from the data by means of standard procedures of multivariate statistical analysis. The choice of variables (*observable* variables may be supplemented with *latent* ones) and specification of their functional relationships are then subordinated to the main goal defined in statistical terms as explaining as largest as possible share of the total variance of each dependent variable. This goal can often be attained – at the cost of making a second generation theory more complicated – by enlarging the list of independent variables and/or trying to express dependence by means of more sophisticated functions.

Theories of the third generation are constructed with the aim of bringing the social sciences closer to the exact natural sciences where the ideals of *parsimony* and *simplicity* need not be sacrificed for the sake of precision or accuracy. Each theory of the kind describes the behavior of a class of *abstract* or ideal systems by means of a small set of *theoretical* variables. Some of them, though not necessarily all, must have their *observable counterparts* in *empirical replicas* of abstract systems. The *idealization strategy* of theory construction in exact sciences rests on distinguishing four categories of *observable* variables operating in each empirical replica of an abstract system: (1) observable counterparts of some theoretical variables; (2) scope variables, or variables whose fixed values are used to state the conditions in which theory-predicted relations between variables of type 1 are expected to hold; (3) disturbing variables responsible for the deviation of actually observed covariances from perfect functional relationships connecting theoretical variables; (4) irrelevant variables, or those recognized to have no bearing on the theory's scope or prediction accuracy.

The interplay between theory formulation and theory testing makes science a going concern. Some research results may prompt a re-classification of variables, for example, a shift of a variable from category 2 to category 1 or 3. If a scope variable is moved to the set of independent variables, the theory will become more general but less parsimonious.

As shown below (Szmatka and Sozański 1994: 230–231), effective use of the idealization strategy encounters serious difficulties in the social sciences, even if the theory's scope is confined to artificial social systems in which the impact of disturbing variables can be reduced to a minimum or at least controlled.

> "In a laboratory system, the experimenter can, to be sure, control the structural conditions of human actions but must always fill positions in the system with concrete individuals shaped in a particular sociocultural context. 'Why is it then that Galileo did not consider the color of his shirt or the phase of the moon when he evaluated the results of his trajectory experiments?' (Willer 1987: 221), and why do sociologists, in order to explain the behavior of experimental subjects, sometimes need to consider such factors as personality or situation variables thought of to be 'at work' in a given setting? 'The answer does not lie in the difference between animate objects which we investigate and the inanimate objects which he investigated. Instead the answer lies in the evidently clean results of his experiments and in the fact that they could be reproduced by him or *by others* as needed.' (*Ibid*.)."

Why *some* empirical sciences are able to produce general and universal, precise and accurate, parsimonious and simple theories? Certainly, the ability to obtain "evidently clean results" in repeated experiments depends to a high degree on how the set of relevant variables is "organized" into a *research design*. The use of suitable mathematical means to express relationships between theoretical variables is no less important, as David Willer observed (1987: 8).

> "In fact, it was through the use of geometric models that the experiments of classical physics were designed. However, the opportunity to systematically generate theoretic models in sociology has required first the development of graph and network theory … Lacking a geometry for the representation of its phenomena, the classical tradition of theory missed the opportunity to develop social theory as formal theory from its outset."

Does the validity of an empirical theory hinge upon the existence of an "order" in the world out there? According to Willer, the "criterion of truth" upon which scientific knowledge is validated is coherence of theory and evidence.

> "Within the process of scientific inference, no assumptions are made concerning the regularity or irregularity of the world. No such assumptions are needed because the relations among objects and events are first drawn in theory and only then compared point by point to bits of information from the world. … Does replication [of experiments] prove that the world is regular? No, for replication proves only that theory can so organize the world and our view of it that at least some parts of our perceptions can be made to appear regular – and that is quite another thing" (Willer 1987: 12–14)

In fact, empirical sciences do without the assumption that the world is regular. Ancient astronomers *did not assume* that celestial bodies behave regularly. They *discovered* that the position of these objects in the sky at any moment can be calculated with great accuracy. The discovery of a "natural order" in some area of the universe may give rise to the form of scientific knowledge, called by Szmatka and me theories of the second generation, but a "hard" scientist, having discovered a regularity, will *try to explain* it by offering a general, universal, precise, testable theory that *abstracts from* particular occurrences of the regularity and concrete objects, which are modeled by ideal objects where "ideal" means "existing within a structured mathematical domain."

Some advocates of the idealization strategy in sociology (Wysieńska, Szmatka 2002) believe that the test of a third generation theory should be conducted within the "theory world" that transcends the concrete "external, phenomenal reality." However, empirical replicas of abstract systems do not differ with the stuff they are made of from empirical systems studied by the theories of the second generation. It is not true that "the social laboratory, unlike the physical laboratory, may be cleanly separated from the phenomenal world outside" (Willer 1987: 214). Willer would be right only if *live* subjects interacting in an artificial computer-aided environment were replaced by computer programs, but *simulating* a theory-predicted process is not equivalent to *testing* the theory. The "theory world" can only be conceived as the world of mathematical domains and set-theoretic constructs. Having entered this world, we can verify logical consistency of a formalized empirical theory, which, once *formulated*, has to be *tested* in the real world, the one we perceive with our senses and rearrange with our actions.

1.2.4. *Theory* and *evidence* should be conceived of as *two* distinct, independent sources of information about *one* world of experience, but *independence* does by no means imply that the data must be "theory free." Even the judicial investigation does not reduce to collecting facts having anything to do with the criminal case. It is "driven" by the prosecutor's theory, which of course may be undergoing modification as new facts are coming to be known. In exact sciences, *relevant* experimental evidence is generated through fundamental or derived measurement. The instruments with which theoretical variables (or rather their empirical realizations) are measured are themselves constructed according to the prescriptions based on the theory being tested. Another tenet of the presented here *common sense philosophy of empirical sciences* states that the two sources of information about the world must not be attributed equal credibility. It is the theory that is to be tested against relevant data, not conversely.

Willer claims that the goal of a *scientific experiment* is to confirm that a theory being tested really "works" – as an effective *instrument* with which we can " organize the world and our view of it" into a coherent whole. Actually, what is being tested is universality and accuracy of theoretical predictions. First, to invoke again Willer's example, we have to ascertain if the motion of a bullet remains unaffected by the circumstances in which it takes place. While universality hardly ever needs to be verified in the physical sciences, in the social sciences, the distinction between relevant and irrelevant variables is not always easy to be drawn; in some cases, one can't ignore "the color of Galileo's shirt" as it can be a disturbing rather than irrelevant variable. Secondly, we need to compare the *observed* trajectory (*recorded* for the bullet by means of an appropriate measurement instrument) with the *theoretical* trajectory (*calculated* for the material point *modeling* the bullet). The theory is considered to have been corroborated if the two trajectories are sufficiently close to each other, the difference between the two being explained by the influence of disturbing variables.

What makes an exact science exact, according to Willer (1987: 220), "is the exact use of theory, not necessarily the exact production of clean results … the criterion should be that a better theory is one which *can* produce cleaner data, not that it always do so.” Obviously, one can't require of any theory that its predictions always be equally accurate. However, a precise theory becomes practically useful insofar as it can provide relatively accurate predictions relatively independently of the context in which it is being applied every time. If very restrictive conditions need to be imposed in order to produce sufficiently “clean” data, then the theory becomes useless outside the setting in which it has passed the test, the setting in which the theory's prediction accuracy has reached the level considered satisfactory in a given discipline. As a consequence, exact sciences are expected to meet another methodological standard besides high precision and accuracy. The results of experimental tests should be *stable*, which means that a small a change of the setting in which a given regularity has been detected in its purest form should cause a relatively small decline in prediction accuracy.

Will the *social* sciences ever attain the level of precision, accuracy and stability comparable to that already achieved in exact *natural* sciences? Physics has always been recognized as the embodiment of the *ideal type* of exact science. Sociology, as it were, called at birth “social physics” (Comte abandoned this name when Quetélet used it to denote the study of *social statistical* regularities), has at best enjoyed (at least since the appearance of Durkheim's *Le Suicide*, 1897) the status of a *normal* empirical discipline.

It is clear that physics and sociology deal with different objects, use different variables to describe them, develop different paradigms and theories, and apply different data generation procedures. Do they also differ with *general* methodology of theory testing? Let us compare a sociologist studying a task group in a laboratory with a physicist investigating the motion of a particle. Both experimenters can trigger off some processes in empirical systems whose behavior is going to be registered, yet the physicist cannot *tell* the particle to move along the theoretically calculated curve, whereas the sociologist, owing to his ability to communicate with human agents, can make them familiar with his theory and induce them to behave accordingly. If we catch a sociologist talking experimental subjects into the behavior predicted by his theory, should we blame him of a violation of a general *methodological norm* or should we rather recognize his *communicative action* as a legal way of testing a *sociological* theory?

If the only purpose of an experiment were to reveal the *form* of a regularity, then it would suffice to *simulate* theoretical behavior in a “virtual world” – where “virtual” means “entirely artificial” rather than “imaginary” or “mental.” A virtual dyadic social system – made up, for instance, of two interacting programs running on two networked computers – is no less real than a pair of human agents who negotiate “consciously” or a mixed system in which a live individual interacts with a computer program. Simulation, no matter how it is technically implemented, cannot replace theory testing. Having seen the theory “in action” in a virtual system, one can never be sure that the theory will work equally well when applied to a real experimental system. Therefore, if we want to learn – what we don't know in advance – if real actors actually behave as regularly as our theory claims, we have to carry out an empirical test. If the experimental subjects “reproduce” the theory because they have been “programmed” to do that by the researcher, then the test turns into simulation.

1.2.5. Testing empirical theories in exact sciences resembles demonstrating consistency of formal theories through constructing their models, that is, the experimenter is looking for an *empirical* domain in which theoretical predictions and relevant observational statements are true. Theoretical predictions, or *empirical consequences* of a formalized empirical theory, are deduced from the formal theory (which was used to formalize the given empirical theory) and some rules linking abstract objects and variables with their observable counterparts.

The formal sciences accept *any* objects for constructing *semantic models*. The set $\{t,m,d\}$ made up of three concrete mortals, we used in the previous section to prove the consistency of our kinship theory, might be replaced with any 3-element set, say, $\{1,2,3\}$. The empirical sciences require the construction material coming from the real world. In addition, my decision to include ordered pairs (t,d) and (m,d) in the empirical relation of parenthood is not arbitrary: I (t) and my wife (m) are in fact parents of our daughter (d).

There is another important difference between *empirical models* and formal semantic models, namely, theoretical propositions – reformulated so as to apply to empirical objects – are usually expected to be merely *approximately* true in appropriate empirical domains. To put it more formally, let $D{:}X\rightarrow\mathbb{R}$ be a variable whose values are assigned to the elements of an empirical domain X by some data generation procedure. Let $T{:}U\rightarrow\mathbb{R}$ be a theoretical variable defined on an abstract domain U. Variables D and $T\circ C$ where C is a mapping of X into U are both defined on X. We call them, respectively, an observable *counterpart* and empirical *representation* of the theoretical variable T. Let r and r' denote the values of D and $T\circ C$ observed and predicted, respectively, for a given object a in X. We say that the *theoretical prediction* $T(C(a))=r'$ has been *corroborated* by the *observational statement* $D(a)=r$, or that the prediction statement is *approximately true* for a, if $|r-r'|\leq\epsilon$ where ϵ stands for the acceptable level of accuracy.

1.2.6. The pattern of theory construction and theory testing which is characteristic of exact sciences will now be illustrated with a sociological example. My aim is also to give the reader a foretaste of the main subject matter of this book, or the mathematical modeling of social systems endowed with a network structure determining the actors' opportunities to negotiate mutually beneficial transactions. Let the empirical domain consist of experimental groups each made up of 5 people who agreed to act according to the following instructions

1. Any group of five subjects is divided into two disjoint subsets, A and B, with 2 and 3 members, respectively. Group members are permitted to communicate with one another solely in couples, each made up of a member of A and a member of B.
2. The communication process in each pair is restricted to negotiating a bilaterally acceptable division of the pool of M points. The two parties are allowed to send offers to each other and respond to each other's offers. If an offer sent by one subject is accepted by the other subject, there follows a transaction, that is, the subjects receive their negotiated shares.

3. Every subject is permitted to conclude no more than one transaction in each negotiation round which ends up as soon as all admissible transactions have been concluded or the time allowed for negotiations expired.
4. All subjects have full information about the system and all events happening there.
5. Every subject should act so as to score as many as possible points for himself in any negotiation round.

Any 5 persons who have understood and acknowledged the above rules become *actors* in an *experimental social system* whose *structure* – determining the actors' opportunities to gain *valued resources* – is established by the experimenter responsible for implementing Conditions 1–3. In virtue of Condition 4, each actor can also watch what is going on in the dyads in which he is not involved. The last condition defines the actors' *goal-orientation*. Each actor is supposed to negotiate so as to maximize his own profit only, but "tactical" decisions on how to pursue this goal are left to himself.

What is the most likely outcome of the systemwide negotiation process? Even though interpersonal differences in bargaining skill may well affect the shape of reward distribution, a *sociological* theory should take into account, in the first place, the *structural* source of unequal benefits. Notice that Condition 1 establishes a 5-person "two-class society" made up of the "upper class" (A) and the "lower class" (B). The two classes differ with the number of transaction opportunities, moreover, in virtue of Condition 3, at most 4 out of 5 actors may gain points in every negotiation round, and the one who fails to find a free partner for a transaction is always a member of B. Hence, each actor in B, in order to avoid dropping out of the game, is forced to make competitive offers to his potential partners in A. Therefore, there are good reasons to suspect that the "rules of the game" give to the members of A an advantage over the members of B. To verify such a prediction, one would have to test the *null hypothesis* H_0: $\mu={}^1/_2M$ against the *alternative hypothesis* H_1: $\mu>{}^1/_2M$, where μ stands for the *expected value* of a *random variable V* whose value $V(x)$ for an experimental group x is the average score of the members of the advantaged class. (We assume that the reader is familiar with the fundamentals of *statistical inference*.).

In order to test H_0, one has to run the experiment with a *sample* of n groups, and compute the *mean* $(1/n)\sum V(x_g)$ of observed values of V. If H_0 gets rejected, the sample mean – used in turn to *estimate* μ – can serve as a measure of how strong is the effect of the "class structure" on "income distribution." Let us add that prior to announcing the discovery of a *statistical regularity*, one has to make sure that the "data points" concentrate around μ and their dispersion can be further reduced by selecting subjects from a more homogenous population.

The ambition of the third type of theorizing in sociology and exact sciences in general is to provide *exact* predictions of certain empirical quantities. Thus, our fictitious study need not end up with decomposing the dynamics of the empirical social system into a constant component (determined by the system's structure) and casual disturbances (produced by the human "material").

Let us represent the communication structure common to all experimental systems as an *undirected* graph $G=(N,L)$ where $N=\{P_i: i=1,\ldots,5\}$ is a finite set whose elements are called *positions*, *nodes*, *vertices*, or *points* of G. L is a subset of $\mathcal{P}_2(N)$, or the set of all 2-point subsets of N. The elements of L are called *lines* (*edges*, *links*) of G. A line $\{P_i,P_j\}$ is a formal counterpart of a *two-way* communication channel connecting two distinct positions P_i and P_j. In accordance with Condition 1, connections should occur exclusively between a subset of N made up of 2 points – label them P_1 and P_2 – and the other subset containing the remaining 3 points. Formally, $L=\{\{P_i,P_j\}: i=1,2; j=3,4,5\}$. Note that the definition of G abstracts from *concrete* technical conditions created by the experimenter to enable communication within a 5-person group.

To build an *experimental* system *modeled* by $G=(N,L)$, one needs to assign actors to positions, which, for any set of 5 subjects, say, {Ann, Bob, Chuck, Dave, Ed}, can be done in 5!=120 ways, for example, $P_1\leftrightarrow$Ann, $P_2\leftrightarrow$Bob, $P_3\leftrightarrow$Chuck, $P_4\leftrightarrow$Dave, $P_5\leftrightarrow$Ed.

> The experimenter may instruct the subjects to associate positions with seats, cubicles or workstations they will be placed at, yet the instructions can be stated without any explicit reference to graph G, its nodes or any other "theoretical" entities. Ann doesn't need to know that she occupies position P_1 connected to positions P_3, P_4 and P_5; she must be instructed that she can communicate with Chuck, Dave and Ed, but not with Bob, etc. The *computer network* is but a means of creating a *social network* with a given communication structure. The program managing an experimental session makes it easier to enforce the rules which define who can contact with whom as well as to record all theoretically relevant events and disable many variables that affect interpersonal communication in a face-to-face setting.

Given a one-to-one correspondence between N and a set of actors, the values of *any* variable defined on N can be transferred from positions to their occupants. Which of many mappings of the set of nodes of G into $\mathbb{R}$ are suitable for building a "structural" theory of social inequality?

Suppose first that the actors in position P_1 have earned on average significantly more points than the occupants of P_2. To account for this finding, we may resort to any variable F such that $F(P_1)>F(P_2)$. Even though F is defined on the set of *theoretical* objects, it is not a *structural* variable because it assumes different values for two positions P_1 and P_2 that are *structurally indistinguishable*, which means that there exists a 1–1 mapping α of N onto N such that $\alpha(P_1)=P_2$ and the image of any line in L is in L (then α is said to *preserve* L, or the *structure* of G). Therefore, our experimental result is an *anomaly* within the general *paradigm* admitting structural variables alone as theoretical predictors of mean earnings on positions. Instead of rejecting the general paradigm, we would rather ascribe the anomaly to a biased actor-position assignment or some flaws in the technical realization of G.

Suppose now that no significant difference has been detected between the earnings of the occupants of P_1 and P_2. As a consequence, we need only one empirical variable V to describe the result of the negotiation process in any experimental system we can represent symbolically as $x_g=(x_{g1},\ldots,x_{g5})$ where x_{gi} stands for the member of gth group placed at position P_i. If only one negotiation round is run for each system and the cases with less than two transactions are excluded from the analysis, then $V(x_g)={}^1/_2(s_{g1}+s_{g2})$ where s_{gi} is the score of actor x_{gi}. Now, instead of contenting ourselves with *estimating* the mean value of V, we can try to *predict* it under the assumption that the division of the profit pool by the occupants of two connected positions P_i and P_j depends solely on the pool size M and the respective values of a *structural* variable F. For example, let us express P_i's theoretical share by the formula $\frac{F(P_i)}{F(P_i)+F(P_j)}M$ and take – as a tentative measure of "structural power" an actor acquires by being placed at P_i – the *degree* Deg(P_i) that is defined in *graph theory* as the number of P_j such that $\{P_i,P_j\}\in L$. Since Deg(P_i)=3, for i=1,2, and Deg(P_i)=2, for i=3,4,5, the predicted gain of an occupant of P_1 or P_2 from a transaction with an occupant of P_3, P_4 or P_5 equals (3/5)M.

To calculate the above theoretical quantity, we assumed that the "power" of an actor is directly proportional to the number of his potential transaction partners. For simplicity, we ignored Condition 3, or the *one-transaction rule*. The rule may generate an even stronger imbalance in payoffs between the two "classes," as it implies that none of the three "applicants" (actors in positions P_3, P_4, P_5) can be sure to get a job when there are only two "employers" (actors in positions P_1 and P_2) who need each to hire just one worker so that one of three potential "employees" will always be "excluded."

The formal theory of "exclusionary power" will be developed in Chapter 3. I have alluded to the theory's "labor market" interpretation (Lovaglia 1999), to reassure the reader-sociologist that the quest for *exact social science* is by no means an escape from classical problems of "social theory."

1.3. Foundations of social science. Social mathematics and mathematical sociology

1.3.1. What unites all mathematical sociologists is the belief that sociology must sooner or later discover the benefits of the use of mathematics, insofar as it aspires to be counted among the basic sciences.

> "The richness … of experience which social science attempts to capture and codify … may be one reason why sociology is perhaps the last of the empirical sciences in which the main stream of effort is as yet almost wholly discursive and nonmathematical. Yet it is the essence of empirical science that, whatever the richness and complexity of the behavior it aims to describe, it must proceed first by analysis into simple regularities, and only then by synthesis into more complex structures. … It is, in fact, the paradoxical combination of simplicity and a potential for expansion into complexity which constitutes much of the value of mathematics as a language for science." (Coleman 1964: 1–2).

Many social theorists doubt about the usefulness of *analytical* tools offered by mathematics because they uphold Comte's teaching on *complication supérieure* of social phenomena. Szmatka and Sozański (1994), who criticized this "foundation myth" of sociology, found quite the opposite view more convincing(Willer 1987: 215).

> "Since the social world is wholly a human construction, it can contain no more information than people have put into it. Since the physical world is not so limited, it is reasonable to infer that the social world contains less information than the physical world and is simpler."

The sociologists who attribute a higher level of complexity to supraindividual entities tend to neglect a more essential difference between social wholes and psychophysical wholes. The former for their fuzzy nature are often harder to identify by the observer within the totality of social phenomena.

If you conceive of complexity as a quality of a cognitive map that is already sketched, then Willer is right. Those known forms of *organized* complexity which exist in the social world are certainly simpler than most organic or nonorganic systems studied by the natural sciences. Nevertheless, the potential for disorder and unpredictability is hardly ever fully actualized in the *real* social world. We know from everyday experience and more systematic historical narratives that people's behaviors form *regular* chains.

1.3.2. The nature of *social regularities* has intrigued early and new "masters of sociological thought." The impact of their philosophical treatises on understanding theory in today's sociology is still strong, as evidenced by the list of books published in the 20th century which were recognized as most influential by 455 members of the International Sociological Association who took part in a survey organized in 1997 (see the ISA website for details). Anthony Giddens, whose book (*The Constitution of Society*, 1984) received 21 votes (rank 14) in the competition which was won with 95 votes by Max Weber's *Economy and Society*, equates regularities with "generalizations," thus agreeing in this respect with the positivist tradition he criticizes for the neglect of human subjectivity and creativity.

> "Some [generalizations] hold because actors themselves know them – in some guise – and apply them in the enactment of what they do. … Other generalizations refer to circumstances, or aspects of circumstances, of which agents are ignorant and which effectively 'act' on them … 'structural sociologists' tend to be interested in the generalizations in this second sense… But the first is just as fundamental to social science as the second … " (Giddens 1984: xix)

Sociologists often content themselves with explaining behavioral patterns occurring in some typical situations by attributing to the actors the *knowledge* of certain *rules* which prompt them what to do in these situations. Giddens (1984: 21–22) defines the "rules of social life" as "techniques or generalizable procedures applied in the enactment/reproduction of social practices." He believes that "awareness of social rules … is at the very core of that 'knowledgeability' which specifically characterizes human agents." Seen in this perspective, the soldiers' obedience results from the knowledge of the rules that establish behavioral dependence between the occupants of inferior and superior positions in social systems of the kind called by Weber *Herrschaftsverband*. "Knowledgeable agents" who happen to occupy positions in such a social system don't need to be instructed explicitly on how to behave, they may well infer the rules from the regular practices they can watch going on around them. No matter how the rules are learned, does the mere knowledge of them suffice to account for the fact that soldiers "as a rule" obey orders of their commanders?

While a "social physicist" would look for a "social force" forcing people to do what they are told to, a sociologist would rather turn to linguistics in search of paradigms making more technical use of the notion of a rule. Weber himself was little interested in language, as evidenced by his *magnum opus* impressing the readers with the author's expertise in law, economy, and religion. His successors in interpretive sociology and students of *discourse* (van Teun 1997 Ed.) have shown more interest in the study of similarities, affinities and connections between social and language structures (Fararo and Butts 1999).

1.3.3. A linguistically informed social scientist will see in any action sequence in a social action system (such as a discussion group or a *Herschaftsverband*) a sample of *parole*, or "speech" in an appropriate "language of social actions" whose"vocabulary" is a repertoire of verbal or nonverbal acts that can be performed by the actors and are recognized by a *verstehende* observer and the actors themselves as *meaningful* in a given interaction setting. Assume that the set of composite expressions that are acceptable in such a *semiotic system* (we will use the term "semiotic code" interchangeably) contains sequences of the form "*B* answers a question asked by *A*" or "*B* does what *A* told *B* to do," that is, two "statements" which form a "string" have been made by *two distinct* "speakers." The occurrence of such sequences admits an explanation in terms of the actors' "grammatical competence" if the notion of a *grammar* is extended so as to cover the rules governing the production of "multi-speaker parole."

In the study of *natural languages*, a single self-contained statement has traditionally remained the uppermost unit of grammatical analysis, yet regular composition of some multi-statement texts justifies the search for syntactic rules on higher levels. As regards artificial formal languages of the kind described in Section 1.1, we owe to the science of *logic* the discovery of the rules that are used by mathematicians and other scientists to produce proposition sequences called *proofs*.

1.3.4. What, in general, is a *rule*? For Giddens (1984: 20–21), "rules are procedures of action, aspects of *praxis*." He recalls Wittgenstein's example from *Philosophical Investigations*. A person *A* (he) writes down a sequence of numbers and asks another person *B* (she) to guess successive numbers that would be obtained if she were to follow the rule that *A* used to start the sequence. Clearly, *B* may try to work out a *general* recursive formula that will enable her to determine for any n the nth term of the sequence when terms from 1st to $(n-1)$th are known. However, the derivation of such a formula – says Giddens – counts as much as it entails the *practical* ability to continue the series of numbers. Any *theoretical* statement of a rule is already an interpretation of it and as such may "alter the form of its application" (1984: 23). The reverse is also possible: A user may deliberately apply a rule in an imprecise way, even if they know very well its strict formulation. No matter whether a rule is under-determined itself or too loosely determines the forms of its application, for Giddens, it still remains an *intersubjectively* identifiable "structural" component of the "encounter" between *A* and *B*.

Peter Winch (1990) has raised doubts as to whether the sciences which accept as evidence the products of "understanding" perception of *meaningful* behaviors can meet the standards of intersubjectivity required of observation in the natural sciences. He assumed that the second person *interprets* the initial sequence and can never be certain of having correctly recognized the rule followed by the first person because the latter is by no means forced to state it so as to help his partner keep the ball rolling. Therefore, neither the observer can assert with certainty that *A* and *B* follow the *same* rule. Giddens does not go that far, as he admits that any "regularly" coordinated coaction may have outside "structural" foundations. However, he immediately adds "the theorem of the duality of structure" which states that "structural properties of social systems are both medium and outcome of the practices they recursively organize" (Giddens 1984: 25). His "structuration theory" is intended to be a third *dialectical* way between "reification" of structure, characteristic of classical "grand theory" (structuralism and functionalism), and the "opposing error of hermeneutic approaches and of various versions of phenomenology, which tend to regard society as the plastic creation of human subjects." (1984: 26).

The new grand theorists who combine dialectics with subjectivism and belief in human creativity find support for their metatheoretical views in historical descriptions of the morphogenesis of concrete sociocultural systems. Indeed, social interaction very often goes on in poorly structured settings in which interpersonal communication begins from a clash of incompatible "definitions of the situation." The emergence of systems of meanings in a social process may become in future the object of third generation theorizing, yet for now social science should keep away from dialectical anti-logic as well as psychologistic or constructivistic interpretations of "subjective meaning" (the Weberian attribute of action) and stick with the traditional paradigm according to which what makes social interaction intersubjectively intelligible and thus scientifically tractable – besides spacetime, physical aspect of coaction – is the actors' *prior* practical knowledge of a *fixed* semiotic code enabling them and competent observers to recognize their actions as meaningful.

1.3.5. "Soft" semiotic systems, seemingly less real than "hard" natural systems, exist within "cultural reality" (Znaniecki, *Cultural Reality*, 1919) which was rediscovered by Karl Popper as the "third world" (Popper, *Objective Knowledge*, 1972). Along with the "first world" (animate or inanimate nature) it admits of intersubjective cognition. Facts from the "second world" (a multitude of individual worlds of subjective experience) are intersubjectively communicable only through their material counterparts in the "first world" or semiotic representations in the "third world." Hence, the sciences which deal with beliefs, emotions, and attitudes must draw on data coming from the first or third world.

So does empirical sociology. Does the "social world" contain elementary facts different from those found in three Popperian worlds? Let us examine the facts described by means of the statements "*A* loves *B*" and "*A* helps *B*." Most if not all sociologists will recognize these facts as *social* as involving two human beings and possibly something that is going on *between* them (Wallace 1988). While biologists tend to identify *A*'s affection for *B* with a state of *A*'s organism, behavioral scientists would rather point out *A*' s behaviors directed to *B* that under a given semiotic code are *signs* of love. But signs of what? Of an *idea* embodied in some behaviors belonging to the first world? Popper allows the presence of Platonic entities in his third world, but he claims at the same time that everything what exists therein is the work of man. Leaving aside the problem of how far human creativity can reach, we can safely assume that men are capable of recognizing directed behaviors as falling under certain categories labeled “love,” “respect,” etc. Clearly, an appropriate criterion must be intersubjectively communicable so that potential users could learn it from those who have designed it. Thus, never ending *hermeneutic* deliberations over the content of concepts and ideas may be left to *arts and humanities*. What the social *sciences* need to have in the third world is only the *term* “love” – the name for a definite class of interpersonal behaviors recognized as “manifestations of love” by those consistently using the term. A behavioral explication of the statement “*A* helps *B*” is even more straightforward.

A's love for *B* – provided that the meaning of “love” is reduced to *expressive* behaviors of *A* oriented toward *B* – seems to be more of a fact in *A*'s private *Lebenswelt*. The second example better illustrates the Weberian concept of *social* action insofar as “*A* helps *B*” means that *A* does something which makes it easier for *B* to achieve her goal. To understand the social fact that consists in a special connection of *purposive* behaviors of *A* and *B*, the observer must take into account not only the actors' subjective intentions but also objective dependence of events: *A*'s action co-determines the result of *B*'s action. Another genuinely social fact, *A*'s power over *B*, can also be described in terms of certain first and second world facts having third world representations. Thus, no matter whether we place “social facts” within the third world, as possibly did Durkheim, or accept the action paradigm proposed by Weber and Znaniecki, we don't need to assume the existence of a specifically *social* stuff.

1.3.6. The natural sciences also need the third world, albeit data collection is based therein on *universal* semiotic systems. While anthropologists must learn to recognize culture-dependent distinctions, for instance, “warm” and “cold” welcome, physicists can rely on transcultural human ability to detect differences in temperature, verbal observational reports being replaced wherever possible by indications of measurement instruments. A recent experiment (reported in *Nature* 450, 2007) has shown that *preverbal* infants are able to recognize helping, hindering and neutral behavior. Thus, there are reasons to believe that the understanding of human acts has equally intersubjective foundations as naturalist observation of physical events.

Nevertheless, natural and social sciences differ with their specific methods for gathering *empirical* information. When a “naive” viewer watches a game being played, *unobtrusive observation* of physical behaviors of the players usually does not suffice to grasp the meaning of ongoing actions. For games like soccer, much of what is seen in the field can be correctly interpreted, provided that innate semiotic competence covers the capability of distinguishing between involuntary and purposive kicks. For more sophisticated games like baseball, even *interactive observation*, or interrogating the players or competent viewers, may fail. This happened to me during my stay in the US when I tried to figure out the rules of this game by consulting the natives. Dissatisfied with their answers that assumed my familiarity with baseball terminology, I turned to a naturalized American who told me that his son born in the US had learned this game by playing it with his peers, that is, through *participant* *observation* – the method the physicists cannot use to explore puzzling practices of elementary particles. Unfortunately, the son could not verbalize the rules in a form intelligible to his father (a mathematician like myself). However, an insider prepared to take the role of an outsider (say, an American student of anthropology) would probably be able to translate practical knowledge of baseball into clearly stated rules.

What does it mean to "follow or apply a rule"? For some rules, such as legal norms or technical instructions, you can't follow a rule unless you know its statement that informs you what to do in a given situation. Rules of the kind, which belong to a piece of social reality under study, should be distinguished from the rules which have been proposed as part of a theory that is to account for observed behavioral regularities. Giddens tends to believe that scientific laws formulated by sociologists do not differ with their epistemic value from the social laws people apply in their everyday activities which are regular and predictable due to the use of these laws. He is right to claim to that *some* social regularities consist in "enacting theories." However, if a theory that is accepted by "knowledgeable agents" is taken by the observer as an explanation of the fact of its being "enacted," then one is forced to abandon even the humanistic Weberian variant of the ideal of nomothetic social science.

1.3.7. The mainstream sociological metatheorizing continues the campaign against positivist and postpositivist (Popper, Lakatos, Toulmin) orientation in philosophy of science. What is being labeled as "positivism" and criticized is both theory-free collecting facts and figures about social life and imitating the patterns of doing theory peculiar to the natural sciences . "In science there is only physics or stamp collecting." If there is some truth in this saying attributed to Rutherford, then radical anti-positivism turns against all attempts to make sociology a normal "positive" science as much nomothetic as possible. Historically, the label of "positivism" has been attached to a number of various meta-scientific view of which some (e.g., the idea of theory-free sense data) deserve to be rejected outright for having little to do with real science. However, few other views, also considered untenable by many leading figures of contemporary social theory, are worthy of defense (Turner 1985). What I mean is the principle of methodological unity of all empirical sciences and that of the demarcation between *empirical* sciences and *hermeneutic* philosophy (Woleński 1989).

In accordance with the first principle, I will regard "direct observational understanding of the subjective meaning of a given act" (*Economy and Society*, Chapter I) as a special kind of *observation*. It's a plausible interpretation of the Weberian concept, provided that "subjective meaning" is not understood too subjectively (as a fact in the Popperian second world) and "meaning" is defined with reference to a given semiotic system.

The demarcation principle does not forbid making *ontological* assumptions in science. The choice of a language for a positive science plays a crucial role in this matter, as it is the language that determines both what "can be said clearly" and about what clear statements can be made. The best way to get rid of existential troubles is to show how entities that are to be studied in a given discipline can be *constructed*. Once this book is to be about *exchange networks*, I have already given (at the end of the previous section) a description of how to construct a social action system of this very special category. To my knowledge, this particular exchange network has not been studied yet in a laboratory, but many other systems of the kind have been. “The taboo against believing in the existence of a social entity is probably most effectively broken by handling this entity experimentally.” I would add to Kurt Lewin's statement, quoted after *Group Dynamics* (2nd edition, 1960, p. 18), that this purpose can also be achieved by locating natural specimens of social entities.“Groups are inevitable and ubiquitous” – we read in the same source (p. 34) – yet of course raison d'être of any nomothetic science may not be high frequency of “natural” occurrences of the entities to be studied.

1.3.8. Let me return again to the question of what are rules. If you analyze various concrete examples, you will arrive at the conclusion that every *rule* is characterized by *conditions of its applicability*, the type of required *input* (the input to an *inference* rule is a finite set of properly structured sentences called *premises*), and the type of intended *output* (in our logical example, a sentence called *conclusion*). To generate the output, which does not need to be known in advance nor uniquely determined, one has to process the input, that is, to perform the operations specified in the statement of a rule. Under such a broad understanding, the term “rule” is the generic term for grammatical rules (in particular, *production rules* in *phrase structure grammars*), rules of games, instructions and regulations, and many other types of rules.

Rules are typical “inhabitants” of the Popperian third world. However given with the “humanistic coefficient” (the term coined by Znaniecki to describe the anchoring of intersubjective “cultural reality” in “human reality”), they can be studied in the absence of their actual or potential users. For example, one doesn't have to interview any voter to examine voting rules. Let us formalize this concept by representing the input to a *voting rule* as any vote configuration $v=(v_1,...,v_n)$, where $v_i=1$ or $v_i=-1$ depending on whether *i*th group member votes for or against a proposal (under this particular formalization, abstentions are counted as votes against). The result of applying the rule to v, or the outcome of the vote, will also be coded as 1 or -1, with 1 now standing for the acceptance of the proposal by the group. The complete formal definition of a voting rule – *constructed* as a *mapping* F of the Cartesian product of n copies of $\{1,-1\}$ into $\{1,-1\}$ – is obtained by imposing on F certain *meaning postulates*.

The first postulate, $F(1,...,1)=1$, states that whenever all group members vote for a proposal, the group as a whole respects their unanimous will. To formulate the second postulate, also formally rendering another a principle of *democratic* decision-making, consider two vote configurations u and v such that $u_i \leq v_i$ for $i=1,...,n$. This relation written as $u \leq v$ means that all who voted for a proposal under u will vote for it under v, so that v may differ from u only in that some of those who did not vote for the proposal under u changed their mind. The second postulate has the form of the implication: if $u \leq v$, then $F(u) \leq F(v)$; in particular, if $F(u)=1$, then $F(v)=1$, that is, a bill that is passed under a given vote configuration will also be passed if it gains additional supporters. To state the last postulate, suppose that the objectors to a bill passed under configuration v want to push through a new bill inconsistent with old one. Since the supporters of the old bill will vote against the new one, the new vote configuration will have the form $-v$, where ith coordinate of $-v$ equals $-v_i$. To make it impossible to pass contradictory laws, F must satisfy the condition: if $F(v)=1$, then $F(-v)=-1$.

These postulates are met, in particular, by the *unanimity rule* ($F(v)=1$ if and only if $v_i=1$ for all i) and the *dictatorial rule* (for some j, $F(v)=v_j$, that is, the will of jth group member always becomes the will of the group). For group size larger than 2, besides these two simplest there exist many voting rules of which the best known is the *simple majority rule* ($F(v)=1$ if and only if $v_1+\cdots+v_n>0$, that is, the supporters of a proposal outnumber its opponents). Suppose that the latter rule is used by an assembly to decide on which of every two out of three candidates x, y, z for an office should be elected if the third resigns. Technically, the votes the group has to count to find a winner in each pair can be collected by asking every member to agree (1) or disagree (−1) with three statements "I prefer x to y," "I prefer y to z," and "I prefer x to z." The group will prefer x to y by virtue of the simple majority rule if there are more voters who prefer x to y than those who don't. Assume that the members of a 3-person assembly vote each in accordance with their own *order* of preference. If the three orders are $x>_1y>_1z$, $y>_2z>_2x$, and $z>_3x>_3y$ ($x>_iy$ stands for "voter i prefers x to y"), then the group prefers x to y and y to z, but contrary to the expectation z is preferred to x. Thus, the collective choice relation determined "democratically" from transitive individual preferences is not always transitive.

1.3.9. David McFarland (see Sørensen 1978: 347) traced back the history of mathematical sociology to the discovery of this fact by Condorcet in 1785. This and all later developments in the *theory of voting* belong to *social mathematics*, a still growing area of mathematics that draws its inspiration from informal social theorizing or methodology of social research. Another example given in the introduction to Part I (equal average number of sexual partners in men and women) also confirms the need to distinguish between social mathematics and *mathematical social science*. The former offers to the latter a range of tools which can be used to produce formalized *empirical* theories. The borderline between these two interactively related fields remains fluid. Thus, many *logically possible* voting rules have so far been analyzed only by the mathematicians. One never knows in advance which of them will ever be implemented and thus will come within the scope of historical and sociological investigations like those voting rules which have already been used by real assemblies.

I used the mathematical theory of voting to illustrate the relationship between mathematics and the social sciences in connection with my recent research (Sozański 2010) inspired by the dispute over designing a voting rule for the Council of Ministers of the enlarged European Union.

Sociologists attached to the old positivist comparative method will look for generalizations concerning historical use of voting procedures. Those having a predilection for experimental method will be more interested in testing certain general and universal hypotheses, say, the one which states that every group is more likely to adopt the dictatorial rule than the simple majority rule if the decisions to be made concern the defense against hostile environment.

1.3.10. Voting rules provide an example in which the output of applying a rule is uniquely determined by the input. For many types of rules, it is not required of the input-output relation to be a mapping. For example, the code of criminal law may allows a judge to choose one from a range of punishments admissible for a given crime. Similarly, in a game like chess (a *two-person game in extensive form*), the player bound to make a move in a nonterminal position *P* is free to decide on which position *P*' will become the next position in the match, his choice being limited to positions immediately reachable from *P* by virtue of the *transition rules*. If *P*' is also a nonterminal position, there comes the turn to act for the other player, and so forth until the game ends with the transition to one of terminal positions to which additional rules assign the outcomes of the game.

If the game ends with *A*'s victory and only a sequence of moves is known, you must be cautious about concluding that *A* has turned out a better strategist than *B*. As occasionally happens in sport games, *A* may well have deliberately played so as to help *B* win. In some social action systems the actors are supposed to pursue certain individual goals and/or a collective goal. The *intentional meaning* of some acts consists in their being recognized by the actors and the observer as *means* to certain *ends*. In other systems, the actors' *goal-orientation*, or their will to bring about definite outcomes of the interaction process, and the *meaning* of their behaviors (directly understandable in the context of a given semiotic code) are separated from each other. Thus, if you notice that *A*'s move gives *B* an opportunity to take *A*'s queen in the next move, you will not say that *A* has broken the rules of chess, but rather interpret *A*'s move as a mistake or an intentional act, a sacrifice to get a strategic advantage over *B* or a help to *B* (if you suspect that *A* wishes *B* to win).

Direct understanding may involve attributing intentions to the actors. For example, an observer of a soccer match, interpreting a kick of the ball, say, as a shot on goal or a pass, takes into account the kicker's intention along with the trajectory of the ball. *Motivational understanding*, called by Weber “explanatory understanding,” is always aimed at revealing subjective motives behind intersubjectively (semiotically) meaningful behaviors or sequences of behaviors. If action chains occurring in a social action system are long and not marked by clear directionality (means-end connections are far from being obvious), a social scientist who wants to understand a given course of action may find it useful to ask the actors themselves to enlighten him about their motives.

Sociologists are divided on the issue of validity and reliability of what the actors present as bona fide reasons for their conduct. Anti-positivists readily rely on of this kind of data, but naturalistically oriented social action theorists, Pareto being their most prominent representative, tend to perceive declared motives as naive or misleading justifications having little to do with real “forces” that make people start or continue their activities (note that, etymologically, “motive” means what causes a motion). While Pareto-sociologist found it necessary to classify these forces into certain types (“residua”), the present “economic approach to human behavior” (Becker 1976, Chapter 1) and related orientation in sociology known as “rational choice theory” (Coleman 1990) avoid handling motivation in terms of substantively defined drives, but rather try to build a general theory of human agency on the sole formal assumption that "Acting man – as Ludwig von Mises put in his treatise on economics (*Human Action.* 3rd revised edition, 1966, p. 13) – is eager to substitute a more satisfactory state of affairs for a less satisfactory. His mind imagines conditions which suit him better, and his action aims at bringing about this desired state." Economic agents who use their subjective preference relations to compare imaginable states of affairs share just one drive that is defined quite generally as the will to be better off. Technically, it is assumed of the actors that they assign numerical utilities to the events in their common environment and seek each as good as possible outcomes for themselves. The utilitarian view of motivation can also be applied to goal-oriented action sequences, provided that the actors are able to assess the costs of actions perceived as means to more distant ends ("Costs are equal to the value attached to the satisfaction which one must forgo in order to attain the end aimed at." Ibid. p. 97).

To formulate and solve various utility maximization problems (see a sociological echo in Part V of Coleman 1990), economics had to employ differential calculus (the branch of mathematics which was once invented to formalize the theory of mechanical motion) and thus became the first social science which admitted mathematics to the heart of mainstream theorizing. Curiously, the author of *Human Action* had a deep distrust of the mathematization trend in economics. His attack (Mises 1966: 350–357) on econometrics (statistical analysis of real life data) and mathematical theories of market equilibrium stemmed from the belief that the necessary condition for applying mathematics in any science are constant relations between quantitative variables, the condition that is not met in the field of human action. Mises and the author of another bible on action (*The Structure of Social Action*, 1937) Talcott Parsons did not anticipate the role a new mathematical discipline, game and decision theory, were to play in the sciences of action.

Nevertheless, one should agree with the masters of nonmathematical economy and sociology that there is no reason to expect from mathematics anything but to supply instruments to describe the shape of a regularity found to exist in a piece of non-mathematical reality. In particular, it is not within the methodological competence of mathematics to render the difference between a pattern that consists in "enacting a theory" by "knowledgeable agents" and a similar pattern arising from the coaction of a set of decision-makers each having limited knowledge of the situation and no intention to co-produce a regular behavioral arrangement.

1.3.11. Giddens' typology of "generalizations" has a counterpart in economics, the opposition between command economy and market economy. In a *market* system, economic agents freely negotiate exchange rates in transactions among one another. The same agents, if for some reasons they have to act in an economic *imperatively coordinated association* (Dahrendorf's translation of Weber's *Herrschaftsverband*), will adopt the exchange rates dictated by the theory they are told to "enact." In a command system, the actors behave "theoretically" for fear that they would be worse off if they did otherwise. In a market system, each of them can improve his own situation through interacting with others, which results in the formation of theoretical (equilibrium) prices. In both systems, the interaction process takes place in a structured environment. In the market case, "freedom of choice" is institutionalized by means of definite rules concerning legal possession, production and exchange of valued resources. “Only within the frame of a social system can a meaning be attached to the term *freedom.*” (Mises 1966: 279).

The truth, rather obvious for sociologists (Willer 1985), about social-structural scope conditions of the laws of market economy has long been overlooked by most economists. “Incredibly, it is only in the 20 of these 200 years [of the history of economics] – as Vernon Smith noted (Smith 1982: 952) – that we have seriously awakened to the hypothesis that property right institutions might be important to the functioning of the pricing system!” Smith demonstrated himself that not only property right institutions do matter. His “experimental handling” of auctions in simple markets has undermined the widespread conviction that economics like astronomy or meteorology has to rely on observation of real-world processes. “My view – wrote Smith (1999: 197) – is that the reason economics was believed to be a nonexperimental science was simply that almost no one tried or cared.”

The Nobel prize for Vernon Smith (2002, with Daniel Kahneman) gave moral support also to the sociologists who like Szmatka and myself tried and cared to do laboratory experiments on exchange, unaware of experimenting that was going on in economics. Experimental research (unknown in turn to economists) on exchange systems with network structure was initiated in sociology at the end of 1970s by Richard Emerson and his collaborators (Cook and Emerson 1978) to be later directed to a new path of development by the Elementary Theory group (Willer 1987; Markovsky, Willer and Patton 1988; Szmatka 1997). Classical economics, which has always focused on *free* exchange of goods and services, has paid little attention to the systems in which social constraints forbid some agents from making some physically possible and mutually beneficial deals. In a *free market*, every two owners of valued resources are allowed to exchange them on the terms that both negotiating parties voluntarily accept. While in such a socioeconomic system the agents' willingness to achieve greater satisfaction remains the only “efficient cause” of any bilateral resource flow, the outcome of the systemwide negotiation process takes a theoretically predictable form if the actors' right (easily alienable as the history of 20th century totalitarian systems has shown) to “pursue happiness” is limited by the prohibition “do not take what is not given” (the Buddhist precept corresponding to “thou shalt not steal”) as well as positively defined by the freedom to give and take (as well as to refuse to take) what others give and to make binding agreements on bilateral resource flows.

In accordance with the *private property rule*, any legal change in the allocation of control over resources can take place only through voluntary give-and-take actions of the actors having each exclusive control over some resource. The economic *reciprocity rule* forces each party of an agreement to give up its resource to the other party as soon as the latter has fulfilled its part of the contract. These rules constitute the *fixed* institutional ground for the functioning of any exchange system. Both free markets and exchange networks can also be endowed with the rules that establish legal ways of negotiating and concluding transactions. Smith (1982) made these rules a *variable* social-structural factor subject to experimental manipulation. Myself, I realized theoretical significance of *negotiation rules* when I repeated (Sozański

1993b) an experiment of David Willer (1987, Chapter 6) to test his predictions as to the functioning of a social system in which a "manager" (in abstract theory language, an actor who occupies the "central" position connected to a set of "peripheral" positions) negotiates with candidates for vacant jobs financial terms of their employment. For the system in which there are as many candidates as vacancies, a reward hierarchy is established by enforcing the rule that the maximum pay the manager may award to the next applicant must be lower than that the pay awarded to the first one. Such a structural constraint makes peripheral actors compete for being first to reach agreement with the central actor, which results in accepting a pretty low pay by the winner of the auction to the benefit of the manager whose payoff depends on curbing payoffs of the employees. Let me quote the conclusion from the English summary of my paper in Polish.

> "[In the replication of Willer's experiment] … the power advantage of the 'center' over the 'peripherals' has been observed, however, to a lesser degree than in the original experiment … the difference can be explained in terms of different modes of negotiating. The rules (imposed by the experimenter or adopted spontaneously by the subjects) which organize the negotiation process can enhance or weaken the competition among peripheral actors."

In my experiment, the "manager" had to hear initial demands of all "applicants" and propose himself the pay for the next person to be hired. Technically, every negotiation round began from a "complete bidding" in which all 7 subjects (6 peripherals and the center) were called (by the computer program) one by one in a random order to present their proposals. Under such a negotiation protocol (Sozański 1993b: 249–250), there appeared "class solidarity" among the peripherals, counterbalancing to some extent within-class competition. While in Willer's experiment the "applicants" went on outbidding one another, in my experiment they often demanded the same pay, accepting the uncertainty of being hired on the terms they defended collectively.

1.3.12. Smith preceded his paper (1982) on experimental microeconomics with the motto (from Agassiz) "Study nature, not books." I studied both, which encouraged me to compare Giddens' bookish metatheorizing with the viewpoint on social regularities that grows out of the practice of experimental research. Our colleagues from the department who practiced "social theory" or historical studies, seeing Jacek Szmatka and me doing experiments on abstract exchange systems, commented on our activities in two ways roughly corresponding to Giddens' two types of social regularities. Some, impressed by detailed instructions we read to our experimental subjects, blamed us of training them to behave theoretically and thus of misconstruing theory testing; others, who took notice of standardized conditions leaving little room for creative "defining the situation," criticized us for treating the subjects like rats, or as Giddens put it, as "agents ignorant of the circumstances which effectively act on them."

It is true that the subjects in any experimental social system are taught and induced to act in accordance with well-defined rules. Once human actions like any other events in the empirical world are subject to the operation of physical, biological, psychological, and possibly social (inter-organism) laws, the sole aim of an experiment may well be to verify if a theoretically designed way of coaction is feasible. Let me call *social praxeology* the empirical science (which should not be confused with Mises' “aprioristic” praxeology) that is to examine *empirical possibility* of implementing *logically possible* systems of social rules. The need for a deeper knowledge of *natural* limits on human creativity in this field is growing because “…the primordial institutions around which societies have developed are being replaced by purposively constructed social organization.”(Coleman 1990: xv).

In network exchange experiments, the enforcement of a social regularity of the first type (negotiation rules and communication channels) is never an end in itself. Apart from establishing definite constraints on and opportunities for negotiating transactions, the experimenter has to arouse in the subjects an appropriate motivation (the desire to earn valued resources) in order to set the “interaction machine” in motion. Once structural and motivational scope conditions of the theory being tested are met, it remains to check if *actual* group process running in the space of *possible* coactions yields theoretically predicted outcomes.

“Structure” alone can never force “agents” to choose definite actions. By saying that a regularity of the second type is produced by the actors themselves, we mean that they try to achieve the *given* “nonstructural” goals, acting within the *given* structures of which they have complete or partial knowledge, and the theoretical pattern, predicted to emerge from their coaction, actually arises whether it is or is not known to those who are producing it. If the subjects come to know the predicted outcomes of their coaction, they may attempt to affect the result of the experiment. How to interpret the case in which the order found to be produced by “naive experimental subjects” does not occur when the experiment is repeated with “knowledgeable agents”? Should we conclude that social regularities of the second type lack the “necessity” that is attributed to the “laws of nature”? Those who know that the processes of market exchange bring about income inequality cannot change the “laws of economy” simply because they do not like some unintended consequences. However, as we know from 20th century history, in some circumstances people can destroy structural and/or motivational scope conditions under which these laws operate. In experiments, the most likely cause of the “knowledge effect” is not knowledge alone but an extra “force” (say, the subjects' ambition to prove their superiority over ignorant rats) that suppresses or interferes with the required motivation.

While it is fairly easy to construct a laboratory social system and enforce conformity with the rules, it is more difficult to make sure that the subjects actually strive to achieve the goals they are enjoined to pursue. For example, if each person reads in the instructions: “Your goal should be to get the best score that you can for yourself through arranging the transactions most favorable to you” (Willer 1987: 121) and a group member is the same time kept informed throughout the negotiation session about the scores of other members, then the observed distribution of earnings may well reflect the subjects' allegiance to an ideal of distributive justice. Indeed, the first sociological experiment on network exchange (Cook and Emerson 1978) confirmed the significance of equity considerations.

1.3.13. An analysis of the foundations of experimental social science leads to the conclusion that more or less complete knowledge the actors have of the environment in which they coact is an essential factor in both types of social regularities. The distinction between *informed* enactment of a theoretical scenario and *informed* coaction producing a behavioral pattern differs from dialectical opposition between proactive knowledge and reactive ignorance. Giddens' conception of human agency apparently departs from the understanding of human action in economics and classical sociological theorizing on social action. According to the traditional point of view that was once labeled by Parsons (1937) the “voluntaristic theory of action” any action involves a choice between two or more behaviors which all make sense in the context of the semiotic “superstructure” of a social action system. The task of the social scientist is then to answer the question of why one of them was actually chosen by the actor. In respect to a regularity of the first type, what is to be explained is why the actors choose to follow rather than break a social rule.

A purely “voluntaristic” explanation of “enacting a theory” assumes that the actors behave in accordance with the theory they know because they *want* it to be true. Having realized that the theory's truth hinges on their voluntary conduct, they act accordingly and construct thereby an empirical model for the theory that describes their most preferred form of collective life. For example, the social order in a Benedictine monastery stems from the monks' will to live together according to the Rule (designed by the founder of the order).

The voluntaristic conception of action takes for granted that the actors make use of their knowledge in pursuing their goals. Knowledge and motivation, however interrelated, should be conceptually separated from each other, similarly as rules and practices; otherwise one would have again to abandon science for dialectics. The actors' convictions as to the nature of social rules play a crucial role in explaining the persistence of theoretically designed social systems. Under the Marxist “activistic” interpretation of command economy, “knowledgeable agents” are assumed to believe that the rules they consciously follow are objective “laws of history.” Both “history makers” (those who claim to have discovered these laws) and ordinary “believers in historical necessity” yield to the “rule” of the rules because they know that any individual breach may only result in negligible disturbances in the order they maintain with their actions. Since the possibility of deviance is not ruled out (otherwise this approach ceases to be “activistic”), one needs to point out the causes for conscious conformity. If we assume that the actors choose to comply with the rules because they are committed to certain values, then we will get a regularity which resembles the one designed by St. Benedict save that unlike the inventors of command economy he perceived his Rule as an implementation of a divine order.

1.3.14. “Value” has always been one of few favorite concepts of sociologists. To define this term, let me first recall that the actors, by assumption, are able to recognize meaningful actions, situations, and results. *Evaluation* is a higher-level semiotic function which presumes *descriptive* semiotic competence. We will speak of a *value* if one of two meaningful categories has been marked as the “positive pole” *preferred* to the other pole by the users of the semiotic code. In order not to rely solely on declared preferences, we add the condition that values must find expression in some behavioral choices sufficiently many actors actually make. For instance, suppose that meaningful verbal behaviors are classified into "telling the truth" and "telling a lie." "Truthfulness" is a value insofar as the plus sign is attached with "telling the truth" and this behavior is observed sufficiently frequently. Nevertheless, once we know that people too often happen to lie, the relationship between values and action needs further clarification.

Let me make a digression here to supplement the methodological characterization of the social sciences. The epistemic rationale for the use of *interview*, which is the main method of data collection in empirical sociology, is the assumption that respondents answering questions asked by the interviewer, tell the truth unless they have some special motives (to be disclosed by the researcher) for doing otherwise.

While the physicists are delighted at the *unity* of the physical world, the social scientists must get to like the plurality of *value codes* and internal diversity characterizing many of them. The physicists need just few kinds of forces (gravitation being the one best known to laymen) to describe the behavior of many physical systems. This can be an uneasy task even for simple systems with just few components (the so called "three-body problem" is an example). In the sciences of human action, even the behavior of a single agent is often very difficult to predict, first of all, whenever two different values prompt different actions in the same choice situation, for example, when an actor chooses between selfish and altruistic behavior in a situation where the benefit of a group member is in conflict with the welfare of a group as a whole. Obviously, the actor must be committed to both values in order to experience a *conflict of values*.

The source of *complication supérieure* of real-world social systems lies in the workings of human motivation rather than in structural complexity of these systems. If the value code has a hierarchical structure (some values are considered "higher" than others), then the actor knows which action one *ought* to choose facing a conflict of values. While moral philosophers readily see in any choice made in such a situation an act of free will, "social physicists" interpret the choice as determined by unequal strength of two conflicting motives, thus eliminating the actor's *will* as a third factor. If two motives are equally strong, it seems quite natural to assume that there is a "judge" whose onerous duty is to resolve the conflict of values. Historians formulate their *ex post* motivational explanations of decisions taken by political leaders in either manner. The task of a social scientist is to *predict* the actor's choice before it is made, which requires that the actor's *value-orientation*, or his actual commitment to the values which make up a given value code, be known in advance.

The actors, by reporting to one another on their evaluations of their own and others' actions, reveal and confirm their attitudes toward values. Their communication becomes then something more than exchange of information on facts. However, making practical choices, especially resolving conflicts of values, is another matter. Heroes are respected and admired but rarely imitated. The acceptance of values declared by the actor should not be equated with his value-orientation conceived as a factor which accounts for his actual behavior.

1.3.15. The form of social order such that the actors' value-orientation perfectly reflects a common value code has always attracted attention of sociologists, as evidenced by the concepts such as Durkheim's "mechanical solidarity" or Weber's "legitimate order." Parsons emphasizes the importance of value consensus for any form of social order. His "sociologistic theorem" reads in Fararo's formulation (Fararo 2001: 97): "A necessary condition for social order is that the ultimate ends of action of the various actors form, to some degree, a common value system."

Does this theorem apply to social action systems in which the actors are *coerced* into following the rules? Before we examine how coercion is related to values, we must recall Weber's fourfold typology of actions (*Economy and Society*, Part I, Chapter I, Section 2). A behavior prompted by an inner impulse or a learned routine is referred to as "affectual" or "traditional" *action* if the actor is aware of the possibility to behave otherwise. The actor can also consider a range of alternative behaviors, compare their anticipated effects and choose an optimal way to attain a given end, in which case his action is called "instrumentally rational." A "value-rational" action has no other end than to confirm the recognition of a value, regardless of the consequences the actor's deed may have for himself and the world. The end that is automatically attained by performing an appropriate action is perceived by the actor as a response to a religious or ethical imperative rather than immediate satisfaction of a mundane desire. Without denying the need for a distinction between "moral" obligations and "natural" drives, I believe that in scientific but humanistic explanations of human action both can be uniformly represented as functionally equivalent motives. Any *motive*, regardless of its organic or nonorganic origin, can be defined as *commitment to a value*, a variable that characterizes *an actor in a situation*.

In many situations, the actor's behavior results from the interplay of two motives working in opposite directions. Conflict of values is inherent in sacrifice and action under coercion. Suppose first that someone devoted to his country has been called in wartime to join the army as a volunteer. His consent is a *sacrifice* if he knows that his refusal will have no other consequence for him than unpleasant awareness of having failed to do his duty for his country. Notice that the dilemma of a volunteer differs from that of a soldier-slave. When the latter hears the order to advance, he knows that a special squad will follow the attacking troop and shoot at those who retreat. Since both alternative responses to the order have equally dangerous consequences, even very weak patriotic motivation is enough to produce obedience. The need to measure the strength of each motive arises only in a situation in which there is a conflict of values.

Imagine in turn a witness in a court of law who received a message that his life is put at risk if he tells the truth. If the witness who values both truthfulness and his own life agrees to give a false testimony, he is said to *act under coercion*. He knows that ignoring the threat may have a deadly consequence for himself, but what will then actually happen will depend on others. He also knows that if he yields to the threat, the only trouble he will have to endure will be the remorse for the betrayal of accepted value. On the one hand, the actor anticipates and evaluates possible *external* events that may follow his action, in particular, possible responses of other actors. On the other hand, the actor realizes that he will feel discomfort or distress if he chooses to act in contradiction with his commitment to a given value. The desire to avoid this unpleasant *inner* experience can be the only motive for compliance with the rules; in such a social system, human monads act "under the starry sky" each according to the "moral law within them." However, sociologists rightly claim that the internal motivational system needs an external underpinning provided by the awareness that a given value orientation is shared by many if not all who approve of a given value code. It is unlikely that a scholar will commit plagiarism if he or she knows that other scholars unanimously condemn this practice, as it were, being an unintended consequence of the devotion to another common value of the academia ("publish or perish"). However, the mere exchange of value judgements does not suffice for the maintenance of value-consensual social order. This kind of social order may not persist unless deviant actions meet with punishing responses of other actors.

A sociologists brought up in the Durheimian tradition will describe a "social fact" of the kind by saying (I will also use this parlance but for stylistic convenience only) that the group punishes its members instead of saying that group members punish other members "in the name of common values" or "on behalf of a group." To define an "action of a group," one needs to specify individual actions of its members and point out the rules the group members apply to coordinate their conduct or assign meaningful results to certain configurations of their actions, as illustrated by an already discussed example (passing a bill by an assembly).

Durkheim referred the concept of "mechanical solidarity" both to large social systems where value-consensual social order rests on the enforcement of "penal law" and to smaller and simpler systems in which the actors comply with the rules for fear of social isolation. It is the threat of being shunned by their peers that can effectively deter scholars from using illegal means to attain the goals valued by their "tribe." Moreno had good reasons to see in positive affective bonds the ultimate basis of social order. People who highly value their interpersonal relations tend to act in accordance with other shared values. The punishment you will suffer when you lose your friend because of your neglect of a value to which he or she is strongly attached differs, to be sure, from a penalty imposed by the court of law, but in either case you are *coerced* to comply with the rules, that is, no matter whether you do or do not consult your "conscience" as to how it would react to noncompliance, you choose the action that will allow you to avoid being punished by other actors.

1.3.16. Any value-consensual social order is sustained by two motivational mechanisms working in concert, internal and external. The role of coercion seems to be relatively greater if the same motives which control the choice of action within the limits determined by the rules defining legal means of attaining goals push the actors into violating these rules . The same greed which makes people seek profit from legal transactions induces the same people to "take what is not given" or break the reciprocity rule, given an opportunity to do it with impunity. While "inner-directed" individuals, who behave as dictated by the value code in which honesty takes priority over self-interest, will resist the temptation to act illegally regardless of how likely is the prospect of being punished, those who have not been trained to constrain their natural desires from inside must face the threat of losing legal opportunities to satisfy these desires. Thus, in maintaining social order in an exchange system, coercion can compensate for the lack of prior *socialization* of actors.

Revolutionary theorists have declared their intention to eliminate coercion in socioeconomic systems they designed as an alternative to free markets, yet as soon as it became clear that some collective values they promoted (such as "fair" allocation of benefits) were unlikely to gain general approval, they resorted to violence to establish and enforce new social rules for production and distribution of material resources. As a consequence, sociologists have to acknowledge the existence of social systems for which the sociologistic theorem does not hold true, that is, compliance with certain social rules is ensured despite disagreement as to fundamental values behind these rules. For example, in an *exchange system with a taxation rule* every actor has to give up a definite part of his profit from a transaction. Some pay the tax because they feel obliged to provide resources for common use, while others contest the tax duty as expropriation and pay solely for fear of punishment.

1.3.17. Any action under coercion, being always a "social action" according to Weber's well known definition, can be called "doubly social" when an actor *B* not only takes account of the *expected* behavior of a concrete actor *C* or unspecified others, but responds to the *actual* behavior of another actor *A* (in particular, *A*=*C*). Many sociologists follow the path taken by Durkheim and Parsons and find it necessary to derive the connections between social actions of *A*, *B*, and *C* from certain tendencies (such as "functional imperatives") which operate on the system level and determine "from above" social regularities on the inter-actor level. Sociology's main subject matter then become "vertical" relations between individuals and a social whole rather than "horizontal" relations in a set of actors mediated by the actors' competence in a common semiotic code.

The power relation is probably the most important *relational* social form. One can agree with Giddens (1984, p. 283) that in social science "There is no more elemental concept than that of power… Power is one of several primary concepts of social science, all clustered around the relations of action and structure." "In its broadest sense, interpersonal power refers to any cause of any change in the behavior of one actor, *B*, which can be attributed to the effect of another actor, *A*." (Zelditch 2000: 1456). In a narrower sense, "power" (*Macht*) was defined by Weber (*Economy and Society*, Part I, Chapter 1, Section 16) as "the probability that one actor within a social relationship will be in a position to carry out his own will despite resistance, regardless of the basis on which this probability rests." Coercion, however not mentioned explicitly, seems to be implied by *B*'s "resistance" that *A* must overcome.

Power is not only most "elemental" concept but most mysterious if it is thought of in terms of the interplay of two wills. Leaving aside the problem of how *A*'s will affects *B*'s will so that *B* actually does what she is told to do by *A*, let us see what can be said about power in a purely behavioral language. First, power differs with its asymmetry from the exchange relation in which either party can initiate interaction leading to a transaction with the other party. The power of a professor *A* over a student *B* means, in particular, that *A* has more occasions than *B* to set a trap for *B* in which *B* has to choose between doing and not doing what *A* wants her to do.

Secondly, the actor who has started interaction has more control over its course, even though he may never be a hundred per cent sure of attaining the intended result because, as Weber stated explicitly however informally, the relationship between the actions of the two parties is probabilistic.

"Probability" was given a more technical meaning by Robert Dahl whose concise definition of power ("*A* has power over *B* to the extent that he can get *B* to do something that *B* would not otherwise do." 1957: 202–203) is no less often quoted in sociological literature. Let *a* and *b* denote, respectively, *A*'s behavior which imposes a choice situation on *B*, and *B*'s behavior preferred by *A* in this situation. Let *a*' and *b*' stand for the actors' alternative behaviors in the respective situations, or any behaviors recognized as failure to do *a* or *b*. If *A* is given sufficiently frequently an opportunity to perform *a*, we could count how often each of two events *A*:*a* and *A*:*a*' (*A* does/does not do *a*) is followed by the event *B*:*b*, and estimate conditional probabilities $p_1=P(B{:}b/A{:}a)$ and $p_2=P(B{:}b/A{:}a')$. *A* has *power* over *B* (Dahl 1957: 204) insofar as $p_1>p_2$; the *amount* of *A*'s power is measured by the difference p_1-p_2. However, positive statistical association of the events *A*:*a* and *B*:*b* is common to various forms of behavioral dependence. Suppose that *B* positively responds to *A*'s call to join the army as a volunteer. Although *B* does "something that she would not otherwise do," we would rather say that *A* has *influence* on *B* rather power over *B*. Hence, Dahl appended three limitations to his *formal* definition of power. First, the event *A*:*a* must precede in time the event *B*:*b*, second, there must be a "connection" between *A* and *B*, but for the sake of generality the meaning of "connection" was deliberately left unspecified. Dahl's third condition turns out most restrictive. While the formal probabilistic criterion does not require that *a* and *b* be anyhow related to each other, in the case of power, action *a* must have the form of a command to do *b* underpinned by a *threat* of punishing *B* for failure to do *b*. For example, a professor orders a student to read a book before the exam and makes her believe that otherwise he will fail her. The student who knows the list of required readings is likely to read the book without being warned against the consequences of ignoring the professor's reminder ($p_2>0$). A significant increase ($p_1-p_2>0$) in the probability that *B* does what *A* wants her to do results from the *threat* that *A* utters or *B* guesses at. Notice that *A* could obtain the same effect by promising *B* to raise her grade on the exam if she reads the book. However, unlike the decision to ignore *A*'s threat, the decision to ignore his *promise* can make *B* worse off than she was before *A* initiated interaction with her. According to Peter Blau (1964: Chapter 5), it is coercion that distinguishes power from other types of behavioral dependence.

B's proneness to obey *A*'s order can be derived from three premises: (1) *B* *believes* that *A* will carry out his threat; (2) *B*'s *prefers* a chance of success to certain failure; (3) *B* applies a rule of *rational* action to (1) and (2). These premises say that *B* is a rational actor having certain preferences and beliefs. Actor *A* appears only in the first condition. His "agency" in the relation with *B* consists in the ability to impose on *B* a choice between *b* and *b*' and to make *B* accept certain claims about the consequences of *B*'s actions. The case where *A* promises *B* a reward for compliance can be analyzed similarly. On the other hand, *A*'s successful attempt to induce *B* to sacrifice something for her country rests of *A*'s ability to modify *B*'s preferences by arousing or enhancing *B*'s commitment to patriotic values.

1.3.18. Thus, our analysis of the ties between social actions in a two-actor system sheds light on how an actor *A* can bring about a definite action of another actor *B*: *A* must be able to create a choice situation for *B* and appropriately shape *B*'s beliefs and/or value orientation. How can *A* convince *B* that his threat is true? More generally, how to make a human being believe that a claim he can't immediately verify is true or at least likely enough to serve as a cognitive basis of action? Biologically oriented behavioral scientists will try to decipher brain structures responsible for cognitive functions. Sociologists will be more interested in how the fact that *B* believes in what *A* has said depends on a social relationship between *A* and *B*.

The "connection" between the professor and his student in Dahl's example provides them with the opportunity to communicate between each other. However, this opportunity is only a prerequisite of *A*'s power over *B*. Communication and behavioral dependence merge together in Wittgenstein's late philosophy – the source of inspiration for the "linguistic turn" in contemporary social theory (Giddens 1984, Introduction). The list of several "language games" we find in *Philosophical Investigations*, remark 23, begins from "Giving orders, and obeying them." More reflections on power appear in remarks 431 and 505.

> "There is a gulf between an order and its execution. It has to be filled by the act of understanding … Must I understand an order before I can act on it? – Certainly, otherwise you wouldn't know what you had to do! – But isn't there in turn a jump from *knowing* to doing?"

In Wittgenstein's philosophical sociolinguistics, "knowing" and "doing" are parts of one communication competence. Weber took for granted that a kind of semiotic community must exist between two actors in order that one of them can impose his will on the other. He would explain *B*'s obedient response to *A*'s order by pointing out – as two possible determinants of the "probability" that an order, once understood, will be obeyed – either actor *A*'s *personal* influence (charisma) or – in the case of "legitimate power" – *A*'s "right" to give commands to *B* and *B*'s "obligation" to execute them, both assigned impersonally to the positions *A* and *B* occupy in an "imperatively coordinated association" (*Herrschaftsverband*). Since we can't observe positions but only some action sequences, we must infer – from what the actors actually do and how they comment on their doings – which of them is the superior and which is the subordinate. Having noticed that the event *B*:*b* always follows the event *A*:*a*, we incline to look at *B*'s behavior (verbal or non verbal) as the result of applying a rule analogous to the grammatical rule "forcing" a speaker of English to put a "direct object" past a "transitive verb." What makes a difference is not the nature of signs that make up the "string" *ab* but its being produced by two different "speakers."

1.3.19. With this remark, our inquiry into the foundations of social science has returned to its starting point – the linguistic conception of social regularities. Certainly, there are many areas of the social world where *semiotic competence* satisfactorily accounts for the course of interaction. For example, if *a* and *b* denote, respectively, a greeting and a conventional response to it, we don't need to employ any "voluntaristic" theory of action to explain the fact that *a* is always followed by *b*. We will simply say that *A* and *B* have mastered the rules of an *etiquette*. Semiotic competence that is required of the users of such a code consists in knowing what to do *and* doing what they know. If for some reasons *a* is more and more rarely followed by *b*, the frequency of *a* will gradually fall, and consequently the social system will stop working. The *semiotic* code that enables *inter-actor communication* within it will also decay as soon as the relations of opposition and complementarity between actions-signs cease to be transparent for the observer for no longer being actualized in *social* interaction.

The social system in which the actors exchange greetings and do not do anything more exists only in the first of "three structural dimensions of social systems: signification, domination and legitimation" (Giddens 1984: 30). "Domination" refers to actual control the actors have over each other's actions and material resources, and "legitimation" to the rights they claim for themselves and grant to each other in interaction situations. Let us assume that a student (*B*) at any face-to-face encounter with a professor (*A*) is supposed to greet him first, while the latter has such an option, both being bound by the etiquette to complete the sequence of greetings initiated by either of them. While semiotics stops at analyzing the relation between *a* and *b* in the action string *ab*, the sociologist, having noticed that the *actors A* and *B* are no longer *interchangeable* in producing *ab*, will explain the fact that the first of two possible sequences of events *B*:*a*, *A*:*b* and *A*:*a*, *B*:*b* happens much more often than the other in terms of some prototypical *positional structure* which exists in the dimension of domination and/or legitimation.

Giddens criticizes (1984: 31–33) imitating *structuralist* lingustics in sociology for the "'retreat into the code' – whence it is difficult or impossible to re-emerge into the world of activity and event." His assertion that "domination … is the very condition of existence of codes of signification" and "the semantic has priority over the semiotic rather than vice versa" sounds reasonable, insofar as it means that the relations between *signs* within a social code reflect rather than determine the relations between *events* produced by the actors in their "world of activity." However, Giddens claims in turn that semiotic representations, including scientific descriptions, of social practices are shaped by "ideologies" which relate "signification to the legitimation of sectional interests." The "linguistic turn" has favored, to be sure, a revival of the Marxian brand of anti-positivism, but the new wave of sociological "grand" theorizing has brought with it earnest attempts to delineate anew the domain of *elementary social facts* and thus provide a conceptual basis for the study of social systems "organized as regularized social practices, sustained in encounters dispersed across time-space" (Giddens 1984: 83). Hence, ordinary sociologists can also benefit from reading the treatises, such as *The Constitution of Society* or *The Social Construction of Reality* (Berger and Luckmann 1966), albeit their own meta-reflection should rather feed on the problems which arise in the context of empirical theories. "Without some concrete laws and principles to ponder – Turner (1985: 24) has aptly noticed – meta-theory becomes … embroiled in unresolvable philosophical issues that are best left to philosophers." Certainly, never ending disputes, like that between the advocates of holism and individualism, should remain the business of philosophers. However, there is one meta-task that any scientist must not cede even to the masters of *analytical* philosophy, namely, the construction of a *language* of which the "limits" will determine the limits of the *world* to be studied in a given discipline. In carrying out this task, one should respect Ockham's maxim ("Entities should not be multiplied beyond necessity"), a natural complement of Wittgenstein's principle ("The limits of my language mean the limits of my world." *Tractatus*, 5.6).

Following these guidelines, I have shown earlier in this section how to build a *minimal* language that enables the description of elementary forms of behavioral dependence. To recap, in order to define power in a dyad, all we need to assume of actors *A* and *B* is that they can: (1) recognize, choose and perform meaningful acts; (2) assess the likelihood of some messages concerning some events possible to happen in their environment; (3) evaluate these events in terms of their desirability for themselves. Thus, the term "will" and whatever it denotes in Weber's definition of power are left beyond the limits of our *scientific* language-world together with few other characteristics of man which appear in sociological *literature*. Lastly, our conceptual model of social interaction was enriched with a positional structure (roughly corresponding to what Berger and Luckmann call "typification" of actors) with the intention to give a nonliterary meaning to any statements of the form: Private *B*, having heard "attention!", stands to attention, provided that he knows that a person *A* who has uttered such a command is an officer. Notice that although both *A* and *B* are physically capable and semiotically competent to give and execute commands of the kind, yet it is *A* that has the right to tell *B* to stand to attention solely by virtue of occupying within a social system a different position than *B*. The power of *A* over *B* in this example is not personal but *structural*. That is to say, one can imagine the existence of an *isomorphic* system in which *A* and *B* swapped places (now *B* is an officer and *A* is a private) without changing their personal qualities described above as (1)–(3). And not only imagine. Arbitrary assignment of positions to actors is part of the construction of any artificial social system with a well-defined range of possible inter-actor encounters.

1.3.20. You can't avoid an inquiry into the *foundations of basic social science* every time a small short-lived self-contained piece of social reality must be created in a laboratory for the purpose of theory testing. The simpler is an experimental system you are to construct, the shorter will be its linguistic-ontological description. Indeed, I could make Chapter 1 much shorter if I chose to confine my metatheoretical analysis to exchange networks instead of building an analytical framework encompassing various types of social systems. I hope that mathematical readers will appreciate the author's restraint. As it were, I did refrain from writing a whole book on "foundations." I also hope that the long introductory chapter I wrote instead will provide nonmathematical readers with enough sociological stuff to think over.

I will close Section 1.3 with few remarks concerning the use of mathematics in the empirical sciences. The distinction – which I have introduced earlier in this section – between social mathematics and mathematical social science can be extended to any empirical science. It was the discovery of non-Euclidean geometries that brought the awareness of the richness of the world of mathematics. Indeed, it contains a great many abstract structured objects, possibly more than will ever be applied as models of real-world objects.

The phrase "*applications* of mathematics in the empirical sciences" seems to convey a sense of one-way dependence of empirical knowledge on the state of mathematical knowledge. Actually, it has often been the case that an empirical theory arose and developed independently until it got ripe for formalization by means of an already available mathematical theory which had been devised without concern about its eventual empirical application. However, it has no less commonly happened throughout the history of the symbiotic relationship between mathematics and empirical sciences that some branches of mathematics have stemmed from the analysis of some simple phenomena. For example, the concept of derivative of a function had its roots in formal defining velocity as a quantitative property of a body in motion. It was physics that gave birth to the *calculus* (also known as *mathematical analysis*) which had been equated with mathematics tout court prior to the emergence of the *theory of sets*, abstract *algebra* and *topology* considered today most fundamental mathematical disciplines.

Kemeny and Snell (1962: 8) once foretold that "one may look forward to the day when the social sciences will be as major a stimulus for the development of new mathematics as physics has been in the past." In fact, when Kemeny and Snell expressed their expectations, the study of social interaction in the dyadic, group or network context had already been stimulating the development of few relatively young mathematical disciplines, such as *game theory* and *graph theory*. For example, mathematical investigations of *tournaments* (a special class of directed graphs; see Chapter 11 in Harary, Norman, and Cartwright 1965) had to do with the discovery of a dominance structure ("pecking order") in animal groups. As I already mentioned in the Preface, my own work on "the mathematics of exchange networks" also began from an attempt to formally describe (Sozański 1993a) the social systems that I saw for the first time in action in my colleague's lab.

While the search for "new mathematics" for the social sciences is going ahead, one should not dismiss various attempts to use "old mathematics" outside its "natural" domain of application. The concept of velocity – once it is formalized as the derivative of an appropriate mapping – loses its specifically physical meaning and becomes applicable wherever there is a need to describe the *rate* at which one of two *continuously* varying quantities changes its value in relation to the other quantity. What must remain particular to each empirical science using differential calculus is only the way in which co-varying quantities are given operational meaning through measurement procedures.

In classical mechanics, the mathematical *form* of the laws of motion is inherently connected with their physical *content*. In the 20th century physics, especially quantum mechanics, the use of mathematics may no longer be described as *formalizing* a pre-mathematical imagery. Whatever can be said clearly about the very existence and nature of elementary particles cannot be said otherwise than in the mathematical language. Hence, to test a theory that deals with the behavior of these mysterious entities, one has to build a long chain of derivations connecting indications of measurement instruments with certain events happening on the level of physical reality where similarity between an empirical object and its mathematical model cannot be assessed through direct perception.

When more weight is attached to mathematical formalism than to data generation procedures, the second pillar of any physical paradigm, the boundaries between physics and other empirical sciences seem to blur. A recent research program for mathematical social science, proposed by a group of interdisciplinarily oriented physicists (Chakrabarti et al. 2006), consists in applying some models of mathematical physics to social processes such as opinion formation within a population whose members are given certain network-determined opportunities to influence one another's views. *Sociophysics* – as this research program has been labeled by its proponents – should not be confused with the reductionist conception of basic social science. *Reductionism* postulates that any form of social interaction which involves communication of two *minds* using a common semiotic system be represented as two coupled *biophysical* processes that are going on in the *brains* of actors *A* and *B*, say, when *A* gives a command to *B*, and *B* hears, understands, and executes it. Sociophysics, like Lewin's *field theory* and other early products of the impact of physics on the social sciences, falls under the second variety of *heuristic naturalism* which recommends that social sciences *imitate* natural sciences in the ways of conceptual and formal representing their subject matter. The imitation usually consists in the search for social *analogs* for physical variables like mass or energy.

The use of mathematical models of physical origin would really make for the unification of social and natural sciences if a given physical paradigm could be imported together with the respective theory, that is to say, the counterparts of the laws operating in a physical domain would remain true in an analogous social domain. To test a sociological theory of the sociophysical origin, one has to devise some measurement techniques for sociophysical variables. Since sociophysics has so far been relying mainly on simulations, it is premature to evaluate this research program in terms of its empirical validity.

1.3.21. The question of whether to construct new mathematical tools or to borrow those already proven from other sciences arises in any empirical discipline at the stage when mathematical modeling becomes indispensable for building theories of the third genus. The use of mathematics in sociology – insofar as the first and second form of theorizing are concerned – can at best consist in: (1) introducing symbolic notation and *logical* rules to formulate theoretical propositions and enable strict deductive reasoning; (2) translating concepts into *variables* and applying to sociological data the rich repertoire of *statistical* methods.

“Statistics is an all-important tool of quantitative sociologists, but the mere practice of statistics for the purposes of estimation and inference – as Sørensen observed (1978: 345) – would not be considered mathematical sociology by most sociologists.” He concluded that the “construction of models implementing a theory about a sociological phenomenon” is at the core of mathematical sociology. I share this widespread view, yet I must add that it is often difficult to draw a sharp boundary between *methodological* and *theoretical* applications of mathematics in sociology. Some procedures of *multivariate* statistical analysis which have long since been known to sociologists (Blalock 1969) require of the user not only to supply a *data* *matrix* (a matrix whose *ij*-entry is the value of *j*th variable for *i*th unit of analysis) to be processed but also to specify *structural equations* that are to describe theoretical causal connections in the set of variables.

What is a *mathematical model*? Whenever you speak of a model, you can be asked to point out something that is modeled by the model. Hence, our question must be phrased more precisely: what is a mathematical model of a nonmathematical entity? A toy model of Eiffel tower is not mathematical because both the tower and its model are material objects, which can also be said about a model and his or her portrait or statue. Interestingly, the word "model" in the second example is referred in ordinary English to the original rather than to its image. In either case, two objects belong to the physical world. They are recognized as similar with respect to certain criteria specified by the user or the constructor of a model. You may insist, for instance, that your souvenir from Paris be made of metal like the original, but you may well define the *modeling relation* as having similar 3-dimensional shape only. Geometric similarity of the toy and the tower can be verified by comparing some lengths and/or angles measured for both things and their constituent components. If a material object has disappeared from the first Popperian world, as it happened to Twin Towers in New York, the construction of an exact copy may still be feasible if a structural-quantitative description of the original is available in the third world.

Many scientists and philosophers expect of mathematics to provide empirical sciences with no more than a *language* which would be better codified than the natural language. Klemens Szaniawski, my first teacher of mathematical sociology, defined (1994: 63) a "mathematical model of an object under study as a set of postulates characterizing that object in the language of mathematics." This usage of the term "model" – he commented – differs from the one that is endemic among the logicians who used to call a model of a given theory any structured domain (a set taken together with some set-theoretic constructs) in which that theory is true.

The expressions "mathematical model" and "formal theory" are often used interchangeably, especially if theoretical propositions are formulated as equations relating variables with one another. I will not conform to this custom that apparently reflects the reluctance of some scientists to acknowledge the *existence* of set-theoretic constructs. Without the distinction between mathematical models and formalized empirical theories, the statement "*t* and *m* are parents of *d*" would have to be referred directly to my family rather than to the set $\{t,m,d\}$ with two relations, which is a model – in the logical sense – of the theory with two primitive relational terms *P* (parenthood) and *E* (sex equivalence) and postulates P1–P6 as axioms. However, the structured set $\{t,m,d\}$, which is a *mathematical object*, need not be juxtaposed only with the formal theory, as it is somehow related to my family too, the latter being an *empirical object*. This relationship, albeit it is hard to describe it quite formally, justifies calling the mathematical object in question a *mathematical model* of my family.

No doubt, my family is something more than the set $\{t,m,d\}$ with two subsets of the Cartesian product $\{t,m,d\}\times\{t,m,d\}$, yet "something more" should not be associated with the holistic creed. What I would like to say is simply that my family and its *mathematical* model – unlike Eiffel tower and its toy model – exist in different ontological domains, and – which can also be said about all material models of material objects – the model does not render everything what is known about the original, in particular, the fact that *t* is a man and *m* and *d* are women.

Similarly, if you focus on the geometric shape of a wooden cube and ignore its weight, type of substance which it was made of, spacetime location and size, you can mathematically represent this object as the set $[0,1]^3$ whose elements are sequences (x_1,x_2,x_3) of real numbers such that $0\leq x_i\leq 1$, i=1,2,3. The points $v_1=(0,0,0)$, $v_2=(0,0,1)$, $v_3=(1,0,1)$, $v_4=(1,0,0)$, $v_5=(1,1,0)$, $v_6=(0,1,0)$, $v_7=(0,1,1)$, $v_8=(1,1,1)$ correspond to 8 vertices of the cube. Its 6 faces and 12 edges are represented as special subsets of $[0,1]^3$; for instance, the image of the edge whose endpoints are mapped into v_1 and v_2 is the set $\{0\}\times\{0\}\times[0,1]$. If you are interested only in how the vertices are *connected* with the edges, you can represent the cube as the undirected graph (V,E), where $V=\{v_1,\ldots,v_8\}$ and $\{v_i,v_j\}\in E$ if and only if v_i and v_j differ with exactly one coordinate. If you forget about numerical coordinates of the points and the geometric nature of inter-point connections, you can use natural numbers (according to Kronecker, it is the only material that God gave to the mathematicians, leaving for them the task of building the rest of the edifice of mathematics) to construct an even more abstract model. The graph (V,E) will then be replaced with the graph (N,L), where $N=\{1,\ldots,8\}$ and $L=\{\{1,2\},\{1,4\},\{1,6\},\{2,3\},\{2,7\},\{3,4\},\{3,8\},\{4,5\},\{5,6\},\{5,8\},\{6,7\},\{7,8\}\}$. You can examine the latter graph without keeping in mind that every a *pair* $\{i,j\}$ in L is somehow related to the *line segment* joining points v_i and v_j in $\mathbb{R}^3$.

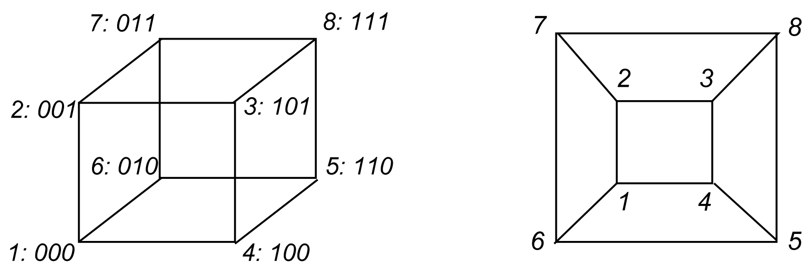


Figure I.1. Mathematical representations of a cube

The term “modeling” is also referred to the reverse operation which yields a visualization of an abstract mathematical construct and helps produce its material embodiment. Thus, any graph can be geometrically represented as a collection of points on the plane and a collection of line segments joining these points as prescribed by the given abstract pattern of connections. Some graphs can be *drawn* in such a way that two segments which have been assigned to any two graph lines have at most one endpoint in common, but not all graphs admit of such a 2-dimensional geometric representation (shown for the graph (N,L) in Figure I.1). An engineer who would try to connect 5 towns with 10 pairwise nonintersecting roads might think that the unfeasibility of this task results from some physical laws, yet the necessity of designing at least one intersection is a simple consequence of a mathematical fact that the complete 5-point abstract graph K_5 does not have a *structural* property called *planarity* (see Harary 1969, Chap. 11). The second small size nonplanar graph is the complete bipartite graph $K_{3,3}$ in which points 1, 2, 3 are all connected to points 4, 5, 6. As Harary stressed in his dedication of *Graph Theory* to Kazimierz Kuratowski, the discoverer of the abstract nature of planarity, this property transcends topology, however it has concrete geometrical and technical implications, as shown by the road network example. There are reasons to suspect that some properties of certain abstract mathematical structures may underlie some "visible" properties of social interaction networks.

1.3.22. Mathematical models differ with the kind of mathematical tools employed. Many areas of mathematics, including traditional calculus and its outgrowths (differential geometry), algebra (group theory, linear algebra), discrete mathematics (graph theory, game theory, theory of formal languages), and probability theory with the theory of stochastic processes, all have already proved useful in the social sciences. Taking account of what is modeled – systems and their fixed structures or phenomena changing over time – static models can be distinguished from dynamic models. These two types of model are supplemented with measurement models which allow for separate operationalization of some theoretical variables in the absence of fundamental measurement.

Berger at al. (1962), who analyzed few typical examples of mathematical models in small-group research, based their tripartite typology on the goals to be achieved through model building. *Explicational* models clarify, refine or generalize original conceptualizations of social phenomena. The objective of any *representational* model is to adequately describe a regularity in a given body of data. *Theoretical-construct* models make use of more abstract mathematical constructs to render hypothetical mechanisms believed to work under the "surface" of directly observable phenomena. An empirical theory formulated with the use of such a model need not fit equally well all relevant data, however it should enable "the theorist to predict the observed process in a wide *variety* of experimental situations" (Berger et al. 1962: 107).

Instead of getting into the issue of taxonomy of models, I will close my remarks on the goals of mathematical modeling with recalling an example that was shown by Thomas Fararo at the beginning of his book (1973: 2–3) on mathematical sociology and social mathematics. His example falls under a particular paradigm for the study of individual performance as dependent on "social" context. It is the "together and apart" paradigm, which has informed a line of research in social psychology since the discovery (1898) of the social facilitation effect. It assumes that the members of a human population can carry out a given task in two contrasting conditions: (1) Each actor works alone without communicating with others; (2) The actors work in n-person teams created by random selection from the population. In the "together" condition, the members of any team, like single actors in the "apart" condition, try each to attain the same *individual* goal, but now they have an opportunity to interact with one another within their team. Under this paradigm, the primary research objective is to find out whether interacting with others facilitates, hinders, or has no effect on goal-attainment. For simple tasks with two outcomes – "success" (the outcome preferred by assumption by every actor) and "failure" (the opposite of "success") – a mathematical reformulation of the paradigm begins from representing any single individual trial as an observation of a value a *random variable* X, which has two values, 1 (success) and 0 (failure). Such a variable is fully characterized by one parameter, the *probability* of success $p=P\{X=1\}$. Groups trials are modeled in turn by means of a sequence $(X_1,\ldots,X_n)$ of 0–1 random variables representing the performance of n distinct persons working together. The statement that n actors act "independently" in the group setting becomes meaningful if the intuitive concept of "independence" is explicated mathematically as *statistical independence* of random variables $X_1,\ldots,X_n$.

Let us define the group failure as the event $\{X_1=0,\ldots,X_n=0\}$, or the joint occurrence of events $\{X_i=0\}$, $i=1,\ldots,n$. Thus, the group fails, by definition, if and only if its every member does. The assumption of independence implies that the probability of group failure is the product of $1-p_i$, for $i=1,\ldots,n$, where p_i is the probability of success of ith group member. The assumption of population *homogeneity* is formalized by means of the condition: $p_i=p$, for some $p>0$ and all i. The two assumptions entail the following formula

$$p_{Gn} = 1 - (1-p)^n.$$

which expresses the probability p_{Gn} of group success as determined uniquely by the probability p of individual success and group size n.

To examine if the above *mathematical* formula renders a *sociological* regularity, we must draw two random samples from the population of humans who understand the task and are able to communicate with each other about it. First, we observe the performance of the subjects working apart. The relative frequency of individual success in the first sample provides us an *estimate* of p that is needed to calculate the predicted value of p_{Gn}. Next, we divide the second sample into pairwise nonintersecting n-element teams that will work together. The relative frequency of group success, observed in the sample which now consists of collective units of analysis, can be used to test the hypothesis which states that the probability of group success equals the predicted value $1-(1-p)^n$. Such a hypothesis can be tested against the hypothesis that p_{Gn} is greater than the latter value or against the hypothesis that it is smaller than it. If the data makes us reject the null hypothesis, our knowledge of "inter-organism behavior regularities" – as Wallace (1988: 31) defined *social* phenomena – has increased too, as we have learnt from the experiment that, for the given n, the *interacting group* copes with the given task better or worse than the *set* of n isolated individuals. One must appreciate every finding of the kind, and even more so its generalizations with respect to the type of task and group size – as long as one can't identify group variables suspected of determining the chance of collective success in the interaction setting.

It is clear that the paradigm just described owes its ability to produce empirical knowledge of some "elementary forms of social behavior" to the *mathematical* explication of *independent* co-action of two or more (human or nonhuman) generators of binary *events*. As we shall see later in this chapter, the notion of "structure" also admits of a formal interpretation, which has a direct bearing on understanding "structural approach" in the social sciences.

1.4. "Structure" and "structuralism" in the social sciences

1.4.1. The many uses of the concept of "structure" in sociological discourse find expression in the bunch of abstract concepts – such as "function," "process," "agency," "content," "disorder" – which are juxtaposed with "structure" as its complements or opposites. Piotr Sztompka (1989), having gleaned a collection of definitions of "structure" or "social structure" from the writings of several leading 20th century sociologists, arrived at the conclusion that various connotations acquired by this concept in sociology can be reduced to four key ideas: (1) interdependence or certain relationships between the parts of a whole; (2) the ideas of order, regularity, repeatability or duration; (3) the existence of a deep, essential dimension, hidden behind the surface of phenomena; (4) the idea of determination, control or influence on empirical processes. I would add to these as (5) the Simmelian idea of form. It may seem to be already covered by (2), but the difference comes out in the comparison of the respective opposite categories. On the one hand, we have disorder, chaos or indeterminacy, on the other hand, content or matter.

In ordinary English, "structure" has both material and formal connotations. According to *Oxford Advanced Learner's Dictionary of Current English* (7th edition, 2005) the noun "structure" means: "(1) the way in which the parts of something are connected together, arranged or organized; a particular arrangement of parts; (2) a thing that is made of several parts, especially a building; (3) the state of being well organized or planned with all parts linked together." Thus, as implied by (2) and (3), "structure," "system," and "organization" in colloquial discourse are often used interchangeably. "Structure" – if conceived as "the *way* in which the parts are arranged" – has a higher by one *logical type* than "a particular arrangement of parts" – under dictionary explication (1) also given as an admissible meaning of the word.

An "arrangement of parts" is usually described more concretely by listing all dyadic connections between some elements. Hence, the attributes "structural" and "relational" often mean the same, as in the introductory chapter of Linton C. Freeman's book *The Development of Social Network Analysis* (Freeman 2004: 2), where we read: "The kind of research that examines the links among the objects is called *structural* … It is present in almost every field of science … In social science, the structural approach that is based on the study of interaction among social actors is called social network analysis." Simmel, who is recognized to have been the forerunner of network structuralism, was first to combine the relational understanding of structure with insistence on abstracting from concrete content of social relations. The essence of Simmel's "formal sociology" is shown in his statement I quote below after Lewis Coser (1977: 180).

> "Geometric abstraction investigates only the spatial forms of bodies, although empirically these forms are given merely as the forms of some material content. Similarly, if society is conceived as interaction among individuals, the description of

> the forms of this interaction is the task of the science of society in its strictest and most essential sense."

The concept of form – Coser comments (1977: 181) – "is freighted with a great deal of philosophical ballast, some of it of a rather dubious nature … Had Simmel used the term *social structure* – which, in a sense, is quite close to his use of *form* – he would have probably encountered less resistance." An explicit explication of structure as form is given in Siegfried F. Nadel's book *The Theory of Social Structure* (Nadel 1957: 7–8).

> "Indicating articulation or arrangement, that is, formal characteristics, structure may be contrasted with *function* (meaning by this term, briefly, adequacy in regard to some stipulated effectiveness) and with *content*, *material* or *qualitative character*. … Thus I can describe the structure of a tetrahedron without mentioning whether it is a crystal, a wooden block, or a soup cube; … This has an important consequence, namely that structure can be *transposed* [italics mine] irrespective of the concrete data manifesting it; differently expressed, the parts composing any structure can vary widely in their concrete character without changing the identity of the structure. Our definition should thus be rephrased as follows: structure indicates an ordered arrangement of parts, which can be treated as transposable, being relatively *invariant* [italics mine], while the parts themselves are variable."

The outstanding Austrian-British anthropologist confused tetrahedron with hexahedron (cube), but he thought of structure-form in terms of "transposability" and "invariance," as if he had a deeper understanding of mathematical structuralism.

1.4.2. The terms *pattern* and *form* are commonly used in linguistics, where *syntactic* patterns in formal or natural languages are conceived as arrangements of empty cells to be filled with lexical content. For example, the learners of English are taught a number of "verb patterns" such as the NVN' pattern (noun+verb+noun). To produce a sentence according to this pattern, one must take three words and arrange them in a sequence, having in mind that the second word must be a verb, while the other two must be nouns. In English, which is an *analytical* language, "Adam loves Barbara" and "Barbara loves Adam" are two different sentences built according to the same pattern in which the order of words is essential. By contrast, in Latin being an *inflectional* language, "Adam amat Barbaram" and "Barbaram amat Adam" are two *stylistic* variants of one statement in which the noun "Barbara" has an ending that marks its *role* in the sentence.

If "Love Adams Barbara" were an acceptable English sentence (semantically, it would tell us that an entity called "Love" acts on Barbara in the way called "Adam"), then the distinction between nouns and verbs wouldn't have to be introduced as part of the description of the NVN' pattern. The pattern, presented in the form SVO (subject+verb+object), would then admit any word in all three "places," S, V and O which stand here for three roles a word can play relative to other words in a sentence. Since in English a word playing role S can also play role O, and conversely, roles S and O can be lumped together to obtain role N (marked also N' to allow for independent substitutions in the pattern NVN'). In linguistics, all words which play the same role, that is, they are *interchangeable* in a class of acceptable statements (*contexts*), are said to be in the *paradigmatic* relationship. The relationships between elements playing different roles in the same context are called *syntagmatic* (see Chapter 2 in Lyons 1968).

Some social interaction patterns can be conceived as patterns in the language in which elementary units are names of *actors* and *actions*. For example, consider the pattern NVN'A and assume that the letter A admits of names of actions as substitutions, while actors' names can be placed in positions labeled N and N'. V is to mark *social* actions such as orders or requests, or those actions which are directed by an actor to another actor and express the former's intention to bring about a definite action of the latter. Let us illustrate this pattern with the sentence |Tom|told|Peter|to-shut-the-door|. Strictly speaking, it is not a string in the symbolic language devised by the analyst, but rather an account of Tom's behavior written down in ordinary English by an observer who saw two persons standing near the door and heard one of them speak “Shut the door, please!” Note that the *verstehende* observer took into account the “subjective meaning,” as the record shows the orientation and intention of Toms' action.

Suppose now that the next record was |Peter|shut-the-door|. Such a statement falls under the pattern N'A. Assuming that the symbols N' and A, which also occur in the NVN'A pattern, are replaced with the same values in either pattern, we can form two complex sequential patterns N'A|NVN'A and NVN'A|N'A. The first of them does not belong to the grammar of the social interaction language because the sequence of actions described by two sentences |Peter|shut-the-door||Tom|told| Peter|to-shut-the-door| could never take place (under the assumption that the actors know what is going on in their common life space). If a linguist found such a record, he would probably conclude that Tom actually said to Peter “Thank you,” but the observer, instead of having written |Tom|thanked|Peter|for-shutting-the-door|, misunderstood Tom's action by classifying it under type V instead of type U containing responses to others' actions. The second pattern NVN'A|N'A yields grammatically admissible sequences of the form “an actor *a* told actor *b* to do something and *b* did what *a* had told him to do.” The study of such sequences goes beyond the scope of traditional linguistics which studies language structures on few levels from *phonemes* at the bottom to *statements* at the top. Under a more general approach, statements are taken as building blocks of *monologs* or *narratives*, texts which are produced by a single speaker, or *dialogs* and *polilogs* in which two or more speakers jointly produce a stream of statements. Dialogs are more interesting for the sociologist because relations within such sequences may reflect *sociolinguistic* relationships or *social* relationships such as power or status hierarchy.

1.4.3. French structuralists have surely done most for the promotion of structure on the market of ideas, yet the term has not become their intellectual property despite their efforts to give it a general while still technical meaning – a meaning that would dictate definite principles of “structural analysis.” Raymond Boudon (1971: 139–140) raised doubts about their claim to the originality of “structural method.”

> “If ‘structural method’ designates a set of procedures for the construction of a theory about any object, with as high a level of verification as possible and permitting one to account for the interdependence of constitutive elements — then we can say that such

> a method does not exist"… there is no 'structural method' in the sense in which there is 'experimental' method."

Boudon concluded that "there are only specific structural theories, some of which are of great scientific importance, while others are less successful." Their common "structural" character, according to him, amounts to the claim that any object amenable to "structural analysis" must be a system. A review of the postulates which have been formulated by various structuralists would probably confirm Boudon's conclusion. However, since the "systems approach" is shared by other general perspectives in the social sciences, in search for more specific principles, especially those admitting of a mathematical interpretation, we must look into the most authoritative texts presenting the scientific program of structuralism (Lévi-Strauss 1968: Chap. 15; Piaget 1971).

Claude Levi-Strauss equates structure with a "model" constructed so as to account for all facts observed within a given whole and to reveal at the same time its systemic character. His further postulate recommends to consider such a model as belonging to a family of models obtained from it by various "transformations." Jean Piaget refers the term "structure" not to an object-model but rather to a system of transformations – conceived as "operations" which can be *composed* with one another. If all are invertible, their set with the "law of composition" is a group in the algebraic sense (to be explained in 1.5.2). For Piaget, it is the investigation of that group that becomes the main task of structural analysis. In mathematics, the term "transformation" is often used to denote a mapping of a fixed set into the same set. In a more "operational" meaning, the term is referred to any rule which can be applied to some objects or states of a fixed object. Unless a transformation is "closed" (cf. Ashby 1956), its scope of applicability and the result of applying to a given "operand" do not need to be uniquely determined nor even known in advance.

Other principles of French structuralism, such as the search for structures lying under the "surface of phenomena" or emphasis on binary "oppositions," seem to me less important, although they have attracted more attention of philosophically oriented commentators. The study of transformations may prove theoretically fruitful – regardless of whether operations are recognized as a product of an autonomous human mind or they are believed to reflect *practical* knowledge people arrive at through rearranging things in the world of their external experience. In spite of little precision and metaphorical language, mathematical thinking imbues the structuralism of Lévi-Strauss and Piaget, especially its "combinatorial" variety which focuses on transformations that do not change the parts of a whole, but only "reshuffle" them to obtain new "configurations" of the same elements. Operations on objects usually are not given in advance, but need to be constructed upon certain knowledge of how an object to be transformed is "structured," that is, one has to know how the parts of a whole are connected with one another. Some transformations are defined so as to leave unchanged the *form* of connection. It is the postulate of studying *invariant* properties of systems, or those properties which are preserved by certain transformations, that seems to be the original feature uniting all varieties of "structural approach" in science.

1.4.4. The *model-transformational structuralism* whose principles have been sketched above has exerted some influence on the study of culture and society. Having contributed to promoting mathematical modeling in the social sciences, it has not taken on within mainstream theoretical sociology. Robert Merton, who embarked on codifying (Merton 1975; Stinchcombe 1977) the principles of *sociological* structural analysis, derived them from sociology's own intellectual tradition, denying any affinity with French structuralism. Indeed, his understanding of *social* structure cannot be placed on the same level of abstractness with such terms as "model" and "transformation," though it admits of a comparison with "structures of social communication" discussed in *Structural Anthropology*. Merton's claim (1975) that the "choice between socially structured alternatives" is central to social structure parallels Lévi-Strauss' view of the choice processes (e.g. the choice of marriage partner in primitive tribes) as determined by "socially established rules." However, for Merton, "the utility or reinforcement of a particular alternative choice" is also "socially established, as part of institutional order." While Lévi-Strauss, speaking of the "information" of marriage rules tends to reduce structural determination to limiting the freedom of choice, Merton apparently ascribes to social structure an endogenous power to force people to achieve "socially established ends."

Piaget (1971) defined "structure" as a "system of transformations" through which a dynamic "whole" is "self-regulating," yet he failed to explain satisfactorily the logic of the interrelationship between the two notions: "transformation" and "self-regulation." According to him (Piaget 1971: 97)

> "By the definition of structure … *all* the social sciences yield structuralist theories since, however different they may be, they are all concerned with social groups and subgroups, that is, with self-regulating transformational totalities. A social group is evidently a whole; being dynamic, it is the seat of transformations; and since one of the basic facts about such groups is that they impose all sorts of constraints and norms (rules), they are self-regulating."

The founders of the model-transformational structuralism claimed universal applicability of their approach. They saw the reason of the resistance to the "structural method" in the sociologists' excessive interest in surface phenomena and ensuing failure to unravel genuine structures governing human actions. "Deep" structures, which remain unrecognized by the actors themselves and hence go unnoticed by those relying solely on people's subjective reports, should not be regarded as metaphysical entities – stressed the structuralists – once every structure is only an abstract model we construct to explain observed regularities. They proclaimed theoretical self-sufficiency of structure and pointed to structural linguistics as the science which provides patterns of theory building for all sciences of culture and society. Certainly, most fundamental language structures can be studied without taking account of particular motives or reasons that are at work every time when the speakers of a given language communicate among themselves. Thus, a grammar reconstructed from a sample of communicative acts has nothing to do with the content of statements and the ends the speakers want to achieve through their communication. By contrast, a social action system can hardly ever be analyzed without reference to the goals pursued by the actors, their values and preferences. Otherwise, as Giddens noticed (see section 1.3.19), there is a danger of “retreat into the code” in explaining social regularities.

Although social systems require a different mode of analysis than language systems, the model-transformational structuralism is not entirely useless to sociology. The idea of invariance directs many theoretical endeavors of sociologists. Sociological structuralism manifests itself, first of all, in the description of human actions in social situations by means of theoretical terms such as “social role.” By saying that a given role, in the simplest theatrical sense, is *invariant*, we simply mean that what an actor playing a role does can be performed by other actors without changing the play. In linguistic terms, the actors who play the same role stand in paradigmatic relation with one another and in syntagmatic relations with actors playing other roles in the same setting.

1.4.5. An “anatomical” view of “structure” and the “physiological” understanding of the twin concept of “function” persist in sociological imagination since Spencer who – according to Leach (1968) – was the first sociologist consciously using the term “social structure.” Structural functionalist paradigm has long prevailed in 20th century sociology, however, after the linguistic turn in social theory, it was overshadowed by new imagery devoid of organic or teleological connotations. In Giddens' structuration theory, “structure” is referred to “rules and resources, recursively implicated in the reproduction of social systems” (Giddens 1984: 377). The remark (“Structure exists only as memory traces, the organic basis of human knowledgeability, and as instantiated in action”) that Giddens appended to his definition shows that the mode of existence he ascribed to any social rule differs from that assumed by Durkheim of “any way of acting (*manière de faire*), whether fixed or not, capable of exerting over the individual and external constraint” (Durkheim 1982: 59). For Durkheim, it is the *contrainte sociale* that grants to any “social fact” (a social rule) an existence independent of “individual manifestations” (instances of applying the rule). The concept of *structural determination* is also found in Marxian thought, albeit its distinguishing feature is rather stress on *structural contradictions*, or dynamic oppositions within and between structures. Simmel's “formal sociology” together with the legacy of social action theorists from Weber to Parsons has been the last but by no means the least important source of ideas for “structural” theorizing in contemporary sociology. In spite of divergent ontological views on the nature of social systems, the omnipresence of “structure” in sociological discourse has hardly ever involved communication difficulties, since one could always rely on the colloquial meaning of the term or invoke a concrete description of how the parts are interrelated to make up a given system.

1.4.6. Although the terms “system” and “structure” happen to be used interchangeably, it is a much more common practice to make a distinction between them, and to speak of the structure *of* a system or of any complex entity. Sometimes “systemness” conveys a richer meaning than being a “whole.” Actors *A* and *B* acting together can be said to form a whole if their actions can be identified within a larger domain of social phenomena as its self-contained “piece.” However, the *A-B* dyad will not be called a system until the actors's behaviors are shown to be *interdependent* in a well defined way, say, the move of player *A* in a game determines the range of possible moves of player *B* who is supposed to act next.

The condition of interdependence is often combined with the requirement that a whole be in a definite relation with its *environment*. The understanding of the latter term varies across theoretical approaches. Ludwig von Bertalanffy, who defined a system as a "set of elements standing in interrelations" (Bertalanffy 1968: 55), conceived of the environment of an "open system" as an outside reality that interacts with the system. Luhmann (1995) stressed that the difference between the system and its environment is an essential part of the system's identity, the environment itself being understood as a substratum over which there arises an area of reduced complexity. An open system preserves or modifies itself the way in which it works and does this through selective mapping the processes running in the external world. The system's environment always remains more indeterminate, diversified or disordered than the system itself. An "autopoietic" system whose functioning involves self-creation determines its boundary and the ties with its environment by itself.

The "systems approach" pervading all contemporary science has found metatheoretical reflection in "general systems theory" (Bertalanffy 1968, Klir 1972 Ed.) – an interdisciplinary project aimed at constructing integrative conceptual foundations for the study of systems of any kind as well as special categories of systems – dealt with by various empirical sciences (physical, technical, organic, ecological, economical, cultural, political, and social systems) or defined in an abstract or even mathematical way. In particular, if every *state of a system* is a collection of values of n variables, then a *system of differential equations* is commonly used to describe how the state variables simultaneously change their values as the time parameter varies continuously.

The rate of change of each state variable is assumed to functionally depend on the current state of the system. Formally, $x'_i(t)=f_i(x_1(t),\dots,x_n(t))$, for $i=1,\dots,n$, where $x_i(t)$ stands for the value of ith variable at moment t and $x'_i(t)$ for the *derivative* of x_i at t.

The term "*social* system" occurs already in Comte's early writings (*Plan de travaux scientifiques nécessaires pour réorganizer la société*, 1822). Today it is associated, first of all, with Talcott Parsons (*The Social System*, 1951; see Parsons 1968 for a summary of his theory) and Niklas Luhmann (*Social Systems*, 1995). Parsons found it impossible or inadequate to define the state of any "social action system" in terms of values of certain quantitative variables. His systems approach – as every analytical framework which addresses a too broad range of phenomena – resists formalization. James S. Coleman (1990), who started like Parsons from "purposive actor," was able to build a formal theory of equilibrium for a narrower but still comprehensive class of social action systems. In his model, the domain of system-relevant social actions consists of inter-actor transmissions of partial control over certain events in the environment.

The relationship between a set of n actors and a set of m events is mathematically represented by two matrices. The *interest* matrix has in the ij entry the relative weight actor i assigns to event j. The control matrix gives the share of the total control over event j currently in possession of actor i. The measures of relative interest in the events are *fixed* parameters characterizing actors. Since the actors may freely pass control over events to each other, the *control* matrix functions in this model as a *variable* state of the system. Given the *initial* distribution of control, the theory's objective is to predict the *final* state of the system – on the assumption that each actor tries to get as much control as possible over the events he is most interested in.

1.4.7. Niklas Luhmann is widely recognized as Parsons' most eminent successor within the system-centered variety of "grand theory." His own theorizing on the nature social systems was informed overwhelmingly by anti-positivist or anti-analytic "continental" philosophy. However, he also adopted or rather adapted for his purposes some ideas which are inseparable from the pro-science orientation within general systems theory. It is, first of all, the concept of "constraint" or "constraint on variety" as defined by W. Ross Ashby (1956: 127). His classical book (*An Introduction Introduction to Cybernetics*) inspired in turn Walter Buckley (1967: 94) to define "social organization" as "a set of common-meaning-based constraints in the ensemble of possible interactions of social units, a reduction in uncertainty of behaviors." Later in the same book (1967: 128), Buckley explained the meaning of "structure" similarly, in line with the custom, prevailing among general systems theorists, of equating "structure" with "organization." Nevertheless, it is "structure" not "organization" that has always stuck in the minds of sociologists as the focus of theoretical debate. As regards the relation between the two terms, one should reserve the latter for a gradable characteristic of complex objects, all having structures of the same type. For example, the fewer dyadic channels provide to an *n*-person group the opportunity of direct or indirect communication, the more "organized" is the group endowed with one of a number of permissible structures modeled each by a symmetric relation such that the respective *n*-point undirected graph is connected. When every group member can directly send a message to any other member, one would say that no "constraint" has been imposed on within-group information flows. However, in this case, there also exists a communication system with a relational structure.

Unlike many leading American sociologists, Luhmann showed more openness to French structuralism and Nadel's "formalism." He recognized the relevance of "form" and "relation," yet it was the notion of "constraint" that he found crucial to the proper understanding of structure, as can be seen from the following statement (Luhmann 1995: 283).

"Systems theory and structuralism agree that structures abstract from the concrete quality of elements. This does not mean that every structure can be materialized in every kind of element but that structures endure despite change in their elements and can be reactualized. ... Precisely for this reason one cannot follow a widely held interpretation and define structures as relations between elements ... Thus structure, whatever else it may be, consists in *how permissible relations are constrained within the system*."

The term "constraint on variety" was introduced by Ashby to denote the relation between the set of actions, system states, events, etc. – *possible* to be observed in a situation – and the smaller set made up of those elements of the larger set that are actually observed by virtue of some *selection* mechanism posited by the observer. To give a familiar sociological example, consider the set of all configurations of responses to *m* stimuli (e.g. yes or no questions), each admitting a "positive" (coded 1) or "negative" response (coded 0). Under the Guttman constraint, the number of configurations drops from 2^m to m+1; for m=3 and some ordering of the stimuli, the predicted configurations have the form 111, 110, 100, 000.

An explanation of the selection mechanism is always a matter of a theory. Such a theory may postulate the existence of some unobservable constructs. Thus, Guttman's theory claims that both the set of stimuli and that of subjects can be mapped into an ordered set *Z*, called "latent continuum" (the mapping of stimuli is assumed to be *injective*, that is, distinct points of *Z* correspond to distinct stimuli), so that subject *i* responds positively to stimulus *j* if and only if $x_i \geq y_j$, where x_i and y_j are the points in *Z* assigned to *i* and *j*.

Luhmann found the concept of structure indispensable for the science of systems only if this concept is taken to mean more than "relations, interdependencies, and invariance." Let me quote again from his major work (Luhmann 1995: 286).

> "All this has the function of a structure only if it is selectively introduced as a constraint on combinatory possibilities. Any further refinement of the concept of structure must therefore be presented as constraint on constraints. ... Therefore we will constrain the concept of structure in another way: not as a special type of stability but by its function of enabling the autopoietic reproduction of the system from one event to the next."

Accordingly, one would have to call "structure" the rule that governs the reproduction of proteins in a living cell rather than reserve this term for any of the configurations which are the input to this rule and consist each of a number of fixed building blocks (of four types: A, C, G, T) arranged into a unique DNA chain.

1.4.8. Our excursion into first generation theorizing in search of "structural insights" ends with the conclusion that thinking of structure in sociology revolves around three concepts, *relation*, *form*, and *constraint*, all on high level of abstractness. I will stay within this conceptual triangle without dwelling any longer on various attempts to further "refine" or restrict the understanding of "structure." This book is to develop mathematical tools for the study of experimental exchange systems, which systems will not be conceived as undergoing "autopoietic reproduction" nor as having internal mechanisms for maintaining stability of the framework that creates both barriers to and room for a variety of social actions.

The concepts of form and relation are key to bringing sociological and mathematical structuralism closer to each other. The notion of constraint may not be left aside, however, as for many sociologists it is an alternative to equating "structure" with any regularity in people's actions – according to Peter Blau's editor's introduction to the collection of papers (Blau 1975 Ed.) by most eminent American sociologists who presented their views on the theme "Focus on Social Struture" of the 69th ASA annual meeting. We find "constraints" also in the first of five "paradigmatic characteristics of structural analysis" proposed by the founder of International Network for Social Network Analysis. Barry Wellman claims (1988: 20) that behavior should be "interpreted in terms of structural constraints on activity, rather than in terms of inner forces within units (e.g. 'socialization to norms')."

The first and foremost principle of network structuralism – in the second place on Wellman's list – gives priority in explaining the behavior of a social system and its parts to inter-element connections which are assumed to be more important than the elements' “inner attributes or essences.” The three remaining postulates are a bit more specific: the search for patterns of interdependence of dyadic ties (interaction processes in dyads *AB* and *AC* are interdependent and dependent on the actions *A*'s two alters in relation to each other); multi-level modeling (structure as “network of networks”); the use of analytical tools that “deal directly with the patterned, relational nature of social structure in order to supplement – and sometimes supplant – mainstream statistical methods that demand independent units of analysis.”

Let me go back to the first “paradigmatic characteristic” of “structural analysis.” “Structural constraints on activity” are taken there to be the opposite of “inner forces” that push the actors to conform to system norms or pursue non-system individual goals. Should we look at structural constraints as “external forces” that press the occupants of the positions in a social system to do something that differs from what they would be doing were they driven only by the “inner forces”? Giddens (1984: 181) quite reasonably rejected such a view he wrongly attributed (see the citation in 1.3.2) to all “structural sociologists.”.

> “Structural constraints do not operate independently of the motives and reasons that agents have for what they do. ... The structural properties of social systems do not act, or ‘act on,’ on anyone like forces of nature to ‘compel’ him or her to behave in any particular way.”

Giddens' theoretical variations on the theme of “structure as constraint” aim at dissolving the Wittgensteinian inter-actor “encounter” in a “societal totality” and equating the “structural” with the “institutional,” “institutions” being understood as “chronically reproduced rules and resources” and “structures” as “rule-resource sets, implicated in the institutional articulation of social systems.” Finally, he defined “structural principles” as “principles of organization of societal totalities” and “structural properties” as “institutional features [of social systems], stretching across time and space” (1984: 375–377). This kind of theorizing fully deserves to be called an “intellectual orgy,” which is the expression Homans referred in the last chapter of *Social Behavior* to his own reflections on the “institutional” and the “subinstitutional.”

1.4.9. Homans' (1974) insistence on the study of *elementary* forms of social interaction agrees with Coleman's view (see 1.3.1) that any basic empirical science should begin from modeling simple regularities. While every mathematical sociologist will readily appreciate discipline and parsimony in constructing conceptual maps of the social world, my sympathy for Homans' methodology does by no means entail acceptance of the language he found most adequate for the description of most elementary social regularities. Nadel's book shows that behavioristic psychology is not the only option for the sociologist who is going to make *action in a situation* the most elementary term of *analytical* sociology. To render the special nature of social phenomena, the observational language must be extended by the notion of a *social norm*. That is, one must assume that all actions or action sequences that are physically and culturally possible in a given situation can be divided into *permitted* and *forbidden*. Given this distinction, two actors will be said to enact the same *role in a situation* if any permitted action or action sequence remains permitted after the actors change places between each other. A *social position* is defined in turn by specifying a set of situations, indicating a role in each, and stating the conditions that have to be met in order that an actor be entitled to enact the given roles in these situations. These conditions can be formulated, in particular, in terms of performing a definite sequence of actions by one or more by interacting individuals. The notion of *social constraints* then becomes a shorthand for saying that definite actions can be "legally" performed only by the occupants of definite social positions and that a relation of dependence exists in the set of positions. A position is said to *depend* on another position if there is a social situation common to the two positions, such that the range of permitted actions for the individual occupying the first position is determined by the action of the occupant of the other position.

The conceptual framework I've sketched above has an essential point of indeterminacy, namely, it has not been said how to introduce the normative division of the set of meaningful actions and meaningful sequences of meaningful actions possible to be observed in a given situation. One can consider three ways of defining norms: (1) A permitted action is that whose frequency is significantly greater then that of the forbidden actions; (2) A permitted action is that whose frequency of acceptance significantly exceeds the frequency of acceptance of forbidden actions; (3) A forbidden action of *A* is that which is followed with sufficiently great frequency by a *negative sanction*, defined simply as a definite action of some *B* or *B*s with some consequences for *A* that *A* would like to avoid. The second definition of a social norm makes sense insofar as the beliefs on which actions are right or wrong – the beliefs the observer can and must learn by communicating with the actors themselves – are included in the theoretical model of an actor. Under the first definition, the observed interaction process and social constraints are indistinguishable. Then, to cite Nadel (1962: 12),

> "We arrive at the structure of a society through abstracting from the concrete population and its behavior the pattern or network (or 'system') of relationships obtaining 'between actors in their capacity of playing roles relative to one another' [the phrase quoted after Parsons]."

Theoretical "abstraction" begins from detaching constant forms of behavior from people through disregarding inter-individual differences in performance. Next, roles are abstracted from situations and a number of relationships are defined in terms of actors' coaction. To classify these relationships into few abstract types and obtain a positional model of a social system, Nadel (1962: 115) proposed two criteria.

> "(i) The first applies to roles between there is no 'dissociation', that is, to roles which we know to involve special relationships with actors in other roles, and which are rendered incomparable only by the qualitative diversity of the relationships. The criterion here is the differential *command over one another's actions*. (ii) The second criterion, though it applies to the first case also, is meant to overcome the 'zones of indeterminacy' in actor-public relationships. In order to do this we reinterpret the 'roles played relative to one another' of individuals so that they have an extraneous reference

point; this can be found in the differential *command over existing benefits or resources*."

These high level structures that the observer must abstract from the actors' actions and co-actions are therefore certain patterns of inter-actor dependence along with the "property structure" which may also be seen as an inter-actor relation – determined by unevenly distributed control over some resources.

In the final part of his insightful book, Nadel concludes that "an orderliness abstracted from behavior cannot guide behavior, resist change, or be passed through by living people." Hence – he continues – we need a "new concept" that "must cover two sets of facts: first, the normative assertions, beliefs, and instructions current in the society in so far as they bear on roles and relationships; and secondly, the institutionalized practices designed to produce and maintain the state of affairs in question" (Nadel 1962: 148). In other words, "structural analysis" – which Nadel considers "to be no more than a descriptive method, however sophisticated, not a piece of explanation" (1962: 151) – must be supplemented by showing some reasons for the very existence and persistence of social constraints. However, since the "second set of facts" – institutionalized external reactions to norm breaking – is only a special part of the stream of all actions observed in a spacetime setting, the state of social order can only be explained ultimately by shared approval of the binding character of certain "normative assertions" (recall the "sociologistic theorem" discussed in 1.3.15).

1.4.10. Metatheoretical reflections on structural method/analysis/approach have led us to the heart of "social theory" – the problem of social order, or the question of why such and not other actions are forbidden in a given social system and why humans who are actors in it conform to the norms, being aware of having an option to behave otherwise. Under Homans' "undersocialized" model of a human being, normatively acceptable behaviors repeatedly happen by virtue of elementary motivational forces working on the subinstitutional level. Parsons' "oversocialized conception of man" (Wrong 1961) posits in turn that all actions of men as occupants of positions in a social system are brought about by social constraints conceived as part of the system's cultural "superstructure." In both cases, the state of social order cannot be distinguished from a particular regular pattern of social relationships that obtains under certain physical and/or social limitations. By contrast, every experimental investigation of a social interaction system (recall the example given in 1.2.6) presumes the distinction between *structure-constraint* – the interaction setting created by the experimenter (his task is to prepare the "stage" and assign positions to the subjects, and if necessary, to impose on them certain rules of co-action) – and the interaction process that is observed under the given constraints, which process is to be analyzed in terms of consistency with a given theoretically predicted *structure-pattern*.

In general, experimental sociology can stay with the formal understanding of "constraint." To study structures-patterns emerging under particular structures-constraints, one needs to assume the existence of a "constraining force" that is located inside or outside the actors and makes them act in accordance with certain rules to achieve culturally defined ends. While "hard" natural constraints can, indeed, be interpreted as *physical* forces acting uniformly on everyone from the outside, "soft" *social* norms narrow down the range of meaningful behaviors through internal motives which may vary across a set of people having to act and co-act in a given situation. Neither the road network (physical constraint) nor the Highway Code (normative constraint) does by itself force the drivers to refrain from illegal behaviors; they must be somehow motivated to obey the rules. They use roads, rather than other surfaces on which wheeled vehicles can move, in accordance with arbitrarily established norms, such as one-way traffic on certain roads, because they are somehow motivated to behave so. They may care about their safety or be afraid of being punished or feel a moral obligation to observe the law. *Motivation* is therefore an indispensable conceptual complement of "structure-constraint."

The distinction between constraints and action under constraints entails the necessity to distinguish between *motivation for conformity* and *extrinsic motivation* that pushes people to pursue certain goals, no matter whether they find an opportunity to do it inside or outside the system. The two functionally different types of motivation need not differ with their intrinsic quality. In experimental exchange systems, the actors want to earn as many points (convertible to money) as possible from transactions among themselves. If they conform to the "rules of the game" because they are remunerated by the experimenter for doing so, their motivation for conformity and their extrinsic motivation have the same "subinstitutional" nature.

It seems reasonable to hypothesize that the structure-pattern of traffic in a highway network will depend to some extent on the structure-constraint of such a system and to some extent on the motives behind the drivers' decisions on from where to where to go and which route to choose. For simple small-size social systems, the extent of "structural effect" can be investigated experimentally, which requires the construction of two or more systems of the same type, with the same number of actors having in either system similar behavior opportunities and motivation, so that the experimental systems differ only in the *form* of their structures-constraints. It is this paradigm that was applied – prior to experimental research on exchange networks – by Alex Bavelas (1950; see also the historical account in Freeman 2004) to examine how the form of network constraint imposed on within-group communication affects group efficiency. To show the logic of "structural approach" of the kind, let me use – instead four 5-actor systems considered by Bavelas – two 3-actor communication systems with network structures displayed in Figure I.2.

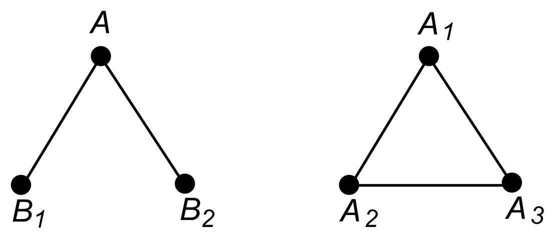


Figure I.2. Two small communication networks (3-point connected graphs)

The task to be done by a team of n persons who are allowed to exchange information under a given communication network is to guess the unique common element of n distinct n-point sets made known each – at the beginning of each experimental trial – to one team member. The n sets to be assigned to n subjects are formed by picking an element s from a set S of n+1 symbols and appending s to all $n-1$-point subsets of $S-\{s\}$. For instance, for n=3 and $S=\{\square,\nabla,\lozenge,\circ\}$, by appending $\circ$ to $\{\square,\nabla\}$, $\{\square,\lozenge\}$, $\{\nabla,\lozenge\}$ we get 3 distinct 3-point sets whose only common element is $\circ$. Notice that the intersection of any 2 out of these 3 sets, e.g. $\{\square,\nabla,\circ\}$ and $\{\square,\lozenge,\circ\}$ has 2 elements (in general, $n-1$ elements), which implies that any team member by sharing with another member his partial information reduces (from 3 to 2 alternatives) the other's uncertainty as to which element is in all sets. The other actor may now transmit his current knowledge to the third actor or wait until the latter passes to him the last piece of information, with the effect that one of them will find the solution.

To analyze the dynamics of collective behavior in such communication systems, one needs to specify acts and norms of co-action, and describe the motivation expected from the actors, as well the assumed level of their knowledge of the structure-constraint and ongoing stream of actions. Our assumptions in this matter are the following: (1) Any system-relevant act consists in sending all information an actor has at a given moment to one of his neighbors in the communication network; (2) No actor can send information to others more than once in each trial; (3) Two transmissions may happen simultaneously only if the respective pairs of actors do not intersect; (4) Every actor knows the network constraint on group action; (5) Any action is known only to the two actors directly involved in it; (6) The problem is recognized by the observer as solved by the group as soon as one of its members discovers the common symbol; (7) All actors want to achieve the collective goal as fast as possible and they don't care who of them will make the last step that yields the solution.

Assumption (3), applied to 3-person groups, excludes simultaneous information flows. Thus, in both systems with network structures displayed in Figure I.2 all transmissions have to be carried out sequentially. The remaining assumptions imply that the chain network (B_1–A–B_2) allows for the following sequences: $(B_1{\to}A)(B_2{\to}A)$, $(B_2{\to}A)(B_1{\to}A)$, $(B_1{\to}A)(A{\to}B_2)$, $(B_2{\to}A)(A{\to}B_1)$, where arrows mark the direction of transmissions. None of 4 predicted strings begins from $A{\to}B_i$ because A (properly speaking, an actor in position A), knowing that B_1 cannot communicate with B_2, must notice that sharing information with B_1 or B_2 will not bring the group closer to the solution, as such an action must be followed by $(B_1{\to}A)(B_2{\to}A)$ or $(B_2{\to}A)(B_1{\to}A)$, thus making the initial action superfluous. In the triangle network, action strings of length 3 cannot be eliminated, however. For example, A_3's action in the sequence $(A_1{\to}A_2)(A_3{\to}A_1)(A_2{\to}A_3)$ is redundant, but A_3 cannot discover it in advance (A_3, unaware of A_1's action, may address A_1 before receiving from A_2 all information sufficient to solve the problem by himself).

Our analysis results in the prediction that a group which works in the triangle network will need on average more time to complete the task than a group placed in the chain network. Clearly, such a hypothesis applies to the systems in which every transmission takes the same amount of time, which condition will be met if all channels in both communication networks have the same technical characteristics. Then the only difference between two systems lies in the form of their network structures. But what should we mean by having the same or different *form*?

> Bavelas anticipated that the readers of his paper (1950) on "communication patterns in task-oriented groups" might understand the concept of "form" in line with traditional 2- or 3-dimensional geometry. To dismiss such an interpretation, he showed two visualizations of one of his 5-node networks, the one with four channels A–B_i. In the first drawing, position A was put in the center with B_1, B_2, B_3, B_4 around it; in the second drawing, A was placed above all B_i, which might suggest a sort of domination over the remaining positions. Bavelas stressed that both geometric representations are equally plausible, however the first *illustrates* the existence of a central position in this network.

Today we know that sameness of form of two "structured" objects can be defined with full precision by the condition that their mathematical models are *isomorphic*. Hence, when we say that two communication networks in our example differ in form, we mean that their mathematical models – two undirected graphs displayed in Figure I.2 – are not isomorphic.

1.4.11. The *relational* understanding of structure pervades the sociology of small groups. Indeed, *microsocial* structures are usually conceived as binary relations (sympathy, respect, power, influence, communication, etc.) in the set of group members. Some relations, such as sympathy, describe a state or process taking place *between* two or engaging two group members. Other relations are derived from comparing values of certain *absolute* variables (Lazarsfeld and Menzel 1964), such as, say, the level of competence in doing something as measured by test scores taken independently for each person. In many areas of social research, especially the study of *macrosocial* phenomena, the term "social structure" is referred to the distributions of a population with respect to such variables. While descriptive statistical analysis of social data focuses on the mere fact that people differ among themselves with values of certain *individual* variables, Blau considers the difference or similarity between A and B in this respect as a hypothetical determinant of interpersonal ties within the population. His theory (Blau 1975, 1977) is based on the notion of a "structural parameter" (1975: 221–222).

> "A structural parameter is any criterion implicit in the social distinctions people make in their social interaction. Age, sex, race, and socioeconomic status illustrate parameters, assuming that such differences actually affect people's role relations. ... The simplest description of social structure is on the basis of one parameter. Thus, we speak of the age structure of a population, the kinship structure of a tribe, the authority structure of an organization, the power structure of a community, and the class structure of a society."

Blau tends to neglect an essential difference between kinship or authority structure, on the one hand, and age, sex or race structure, on the other hand. Kinship *relations* underlie kinship *roles*, not conversely (A's status as a parent results from being in the parenthood relation with some B). Age roles come from a partition of a population into age categories, say, children, adolescents, adults, and the elderly. Since these categories are ordered, there exists a relation in this case too. Sex or race structure, which is defined in terms of the values of a "nominal" rather than "graduated" parameter, can also be seen as generated by an equivalence relation (recall 1.1.4). However, Blau's theory of homophily (people who have the same or close values of structural parameters of either type are more likely to make friends) has been subsumed under social network analysis because of the intrinsically relational nature of the theory's *explanandum,* a non-random arrangement of positive interpersonal ties in a non-homogenous population.

If a scientific theory is conceived as an organized set of propositions which relate some variables to one another, then it is the use of structural variables that is the distinctive feature of structural or structuralist theories. The meaning of the *adjective* "structural" seems to be secondary to that of the *noun* "structure." Nonetheless, we find it preferable to leave aside deliberations on what should be meant by the "structure of a whole or system" and try to define "structural variables" – as a special subset of the set of all variables that are to theoretically describe a given class of wholes. Lazarsfeld and Menzel (1964: 427–428) distinguished three types of "collective properties." "Analytical properties" are "obtained by performing some mathematical operation upon some property of a single member," "properties of members" (1964: 431–433) being divided in turn into four classes: "absolute, relational, comparative, and contextual." "Structural properties" are "obtained by performing some operation on data about the relations of each member to some or all of the others." "Global properties" form the third residual category in this typology.

The attribute "structural" – when referred to a whole – often has relational connotations, as in Lazarsfeld and Menzel's explicit statement. However, it is also used to denote any permanent characteristics of a system regardless of their relational or non-relational nature. In economics, fixed "structural" conditions persisting over time are contrasted with "conjunctural" fluctuations. Lastly, in multivariate analysis there is a notion of "structural equations" with the qualifier "structural" is now referred not to variables but to functional ties among them.

In the next section, we show that structure, conceived of in mathematics as something that is *constructed* from the elements of a given set, has there the twin notion that serves to compare "structured" entities and clarify the connection between the concepts of "structure" and "form." I mean the notion of isomorphism (etymologically, sameness of form, Greek *morphe*), which can also be used to define "structural properties" of mathematical objects as the properties that are "invariant," or shared by isomorphic objects.

1.5. "Structure" and "structuralism" in mathematics

1.5.1. In mathematics, structure has fairly recently come into prominence, although the need to introduce such a term in a more technical but still very general meaning has been vaguely felt since more or less the middle of the 19th century. Inspired mainly by the rapid development of algebra (Corry 2004), the mathematicians faced the problem of unity of their discipline when it became clear that the mathematical universe extends far beyond familiar numerical domains and geometrical spaces known to everybody from everyday experience. The discovery of antinomies resulting from too free use of the concept of a *set* gave rise to intense investigations of the foundations of mathematics. Along with the birth of metamathematics there emerged a demand for such a definition of the field of mathematical research that could satisfy working mathematicians regardless of their interest in meta-problems. Such a "positive" or "positivistic" image of mathematics was offered in the voluminous treatise *Eléments de mathématique* published by the team of French scholars who used the nickname Nicolas Bourbaki. Their point of view was marked in the title of their work, as Bourbaki chose to speak of *la mathématique* instead of *les mathématiques* (the plural) as in ordinary French.

The double, methodological and substantive, unity of mathematics finds expression in two fundamental notions of "proof" and "structure." The deductive method has worked in mathematics since Euclid in an essentially unchanged manner, only new tools for formalizing proofs were being invented and the view on epistemological status of axioms changed in that it was no longer required of the basis for deducing mathematical theorems to contain only self-evident truths. While an axiomatics can be arbitrarily chosen, the only concern that remains is to ascertain consistency of the mathematical theory that is deduced from the axioms.

The general notion of structure appeared in the middle of 20th century. It was for the first time defined precisely in the first volume of Bourbaki's treatise (1957; English edition, *Theory of Sets*, 1968). Many particular types of structures, to be sure, were known much earlier but to formulate a generalizing definition was apparently not an easy task.

I shall present here the elements of this definition in a manner, I hope, sufficiently precise for mathematical readers but not quite formal, as too much formalism has made Bourbaki's work difficult to read even for mathematicians. Bourbaki starts from defining a *species of structures* Σ as a text that makes sense within the context of a particular mathematical theory T (containing the *theory of sets*). The text Σ provides a formal description of how to *construct* any structure s of the species to be defined. First, one must point out a finite number of *principal* and *auxiliary base sets* – as the "building materials" for the construction. At least one principal set is always needed, while auxiliary sets, which may be absent, remain a fixed component of the procedure applicable to variable principal sets. Secondly, one needs a prescription of how to construct the set whose elements will be structures of the given species, that is, one must state what actions and in what order have to be performed on the base sets and sets obtained from them at each successive step of the construction. The proposition which states that s is an element of the set obtained at the end is called the *typical characterization* of the species of structures Σ. The two set-theoretic operations that Bourbaki considered sufficient for constructing all species of structure are: forming the Cartesian product of two sets and taking the set of all subsets of a set.

Thirdly, the definition of a given species of structure may include *axioms*, or certain conditions to be met by the base sets and s constructed from them. The double role of axioms consists in that they provide the starting point for deducing theorems and at the same time delimit the class of objects which the theory applies to. The theory obtained by adding to T the axioms and the typical characterization of Σ is called the theory of the species of structure Σ.

Assume now, for the sake of simplicity, that only one base set X is used. We say that S is a *structure* of the species Σ on a fixed set E, or that E is *endowed with a structure* S of the species Σ, if S has the required typical characterization and E and S satisfy the axioms of Σ. The ordered pair (E,S) is called the *mathematical object* over the set E with structure S.

1.5.2. We give now some examples of species of structure (read 1.1.5 again to recall few elementary set-theoretic terms that re-appear below). Let the statement $r\in\mathcal{P}(X\times X)$ be the typical characterization for the species of structures to be defined. That is, r is an element of the set constructed by first forming the Cartesian product $X\times X$ and next taking the set of its subsets $\mathcal{P}(X\times X)$. A structure R on E of such a species is called a *binary relation* on the set E. The mathematical object (E,R) is called a *directed graph* (or, briefly, a *digraph*) with the *point set E* and the *arc set R*. We will consider later on only *finite* digraphs defined by one axiom which states that X is a finite set.

Binary relations are just one kind of structures. Two other important kinds are *algebraic* structures and *topological* structures, dealt with by algebra and topology being two basic disciplines that provide most fundamental terms for building other parts of the "edifice of mathematics."

The key term of algebra, a binary *operation* on a set X, is defined as any mapping of the product $X\times X$ into X. Thus, the typical characterization has now the form $s\in\mathcal{P}((X\times X)\times X)$ and the first axiom, common to all particular algebraic species of structure, states that s, or a relation between $X\times X$ and X, is a mapping, that is, to *any* ordered pair (x,x') there corresponds one and only one element x'' in X.

Let $\circ$ denote an operation on E. Its result for an ordered pair (e,f) of elements of E will be written $e\circ f$ instead of $\circ(e,f)$. If $e\circ f=f\circ e$, for any $e,f \in E$, the operation $\circ$ is said to be *commutative*. It is said to be *associative* if $e\circ(f\circ g)=(e\circ f)\circ g$, for any $e,f,g \in E$. An element $j\in E$ such that $j\circ e=e\circ j=e$, for any $e\in E$, is called a *neutral element* or a *unit*; every associative operation may admit of only one element with this property. Any set E endowed with an associative operation having a neutral element is called a *semigroup* and a *group* if it is assumed besides these axioms that every element e of E has an *inverse*: an element e' in E such that $e'\circ e=e\circ e'=j$. By associativity, the inverse of e is unique; it is written e^{-1} or $-e$ (for commutative operations).

To give an example, let us take as E the set of all permutations of a fixed set A (*permutations* are 1–1 mappings of A onto itself) and endow E with the operation that is defined as the composition of mappings. The unit for this operation is the *identity mapping* i_A: $i_A(a)=a$, for any $a\in A$. Another group, probably best known to the people who equate mathematics with numerical calculations, is the set of integers $\mathbb{I}$ with addition as operation and 0 as neutral element. An even simpler example of a *commutative group* is the set {0,1} endowed with the operation + defined by the formulas: 1+1=0+0=0, 1+0=0+1=1. The elements 0 and 1 are also written as signs + (plus) and − (minus), respectively. Under the multiplicative notation (used instead of the additive one that is customarily preferred for commutative operations) the formulas defining the group operation are: $+\cdot+ = -\cdot- = +$, $+\cdot- = -\cdot+ = -$.

1.5.3. Group is one of few pivotal concepts of modern mathematics. The importance of this particular species of structure lies not only in that groups are worth studying in themselves but even more in that the theory of groups is a tool for analyzing all species of structure, namely, for any mathematical object $O=(E.S)$, you can define a group whose elements are those 1–1 mappings of E onto itself which "preserve structure" S. Such mappings are called "automorphisms" or "symmetries" (see Weyl 1980) of the given mathematical object. The *automorphism group* Aut(O), also called the *symmetry group* of O, is a *subgroup* of the permutation group from which it "inherits" the operation owing to the fact that the composition of two automorphisms is an automorphism and so is the inverse of any automorphism.

An *automorphism* of (E,S) is defined as an isomorphism of the object with itself. To explain what is meant by an isomorphism of two mathematical objects (E,S) and (E',S') with structures of the same species Σ, let us say first that the term "isomorphism" will be referred to any bijective mapping φ of E onto E' such that the correspondence between elements of E and E' induces a correspondence between the structures S and S' in these sets. The meaning of the latter correspondence will become clear when we define the *structure transported from* (E,S) *to a set* E' *by a bijective mapping* φ *of* E *onto* E' (in particular, $E=E'$) – as a structure (of the same species, that is, having the same typical characterization and satisfying with E' the same axioms) that is constructed from S with the use of φ. How does the transported structure – it is noted $\varphi(S)$ and also known as the *image* of S through φ – depend on S and φ? I will not try to provide a general formulation that would fit any typical characterization and hence would be too complicated. Instead, let me show a simple example. Let (E,R) be a directed graph. The relation transported from (E,R) to E' through φ, or the *image* $\varphi(R)$ of R through φ, is the subset of $E'\times E'$ which consists of all ordered pairs of the form $(\varphi(e),\varphi(f))$, for any $(e,f)\in R$.

Two mathematical objects (E,S) and (E',S') with structures of the same species are said to be *isomorphic* if there exists a bijective mapping φ of E onto E' such that $S'=\varphi(S)$; the mapping φ is then called an *isomorphism* of (E,S) and (E',S'). Thus, two digraphs (E,R) and (E',R') are isomorphic if and only if there exists a bijection φ such that $R'=\varphi(R)$, that is, $(\varphi(e),\varphi(f))\in R'$ for any $(e,f)\in R$ and for any $(e',f')\in R'$, $e'=\varphi(e)$ and $f'=\varphi(f)$ for some e,f such that $(e,f)\in R$.

Two groups, E with operation $\circ$ and E' with operation $\circ'$ are isomorphic through φ if and only if $\varphi(e\circ f)=\varphi(e)\circ'\varphi(f)$ for any e,f in E. For example, the *exponential function* with the base p (p is a fixed real number such that $p>0$ and $p\neq 1$) – it assigns to any real number x the number $y=p^x>0$ – is a bijective mapping of the set of all real numbers onto the set of real numbers greater than 0. The first set and the second set, endowed, respectively, with addition and multiplication of real numbers, are groups. The exponential mapping is an isomorphism of these groups in virtue of the formula $p^{x+y}=p^xp^y$. The *inverse isomorphism*, the mapping of $(0,\infty)$ onto $(-\infty,\infty)$ given by the formula $x=\log_p y$, is known as the *logarithmic function*. The formula $p^0=1$ illustrates the fact that any isomorphism involves a correspondence between "special" elements of two mathematical objects..

1.5.4. We consider now the species of structure of which the typical characterization is $s\in\mathcal{P}(\mathcal{P}(X))$ or equivalently $s\subset\mathcal{P}(X)$. A mathematical object (E,H)

over the base set E with structure H is called a *hypergraph*. The image of a family H of subsets of E through φ consists of those subsets of E' which can be represented in the form $\varphi(A)=\{\varphi(a): a\in A\}$, for all A in H. This species of structure has 3 important *subspecies*, each determined by certain axioms imposed on H, the respective mathematical objects being called graphs, voting games, and topological spaces.

The structure W of a *voting game* (E, W) with *assembly* of voters E is a family of subsets of E – called *winning coalitions* – that satisfies the following axioms: (1) $W\neq\varnothing$ (there exists at least one winning coalition); (2) For any C, $C'\subset E$ such that $C\subset C'$, if $C\in W$, then $C'\in W$, that is, any set of voters containing a winning coalition is also a winning coalition; (3) For any $C\subset E$, if $C\in W$, then $E-C\notin W$. The third condition, which means that the non-members of winning coalition do not form a winning coalition, makes it impossible to pass simultaneously two contradictory bills, one supported by the members of C and the other supported by the members of $E-C$.

Graphs are hypergraphs such that every element in H has 1 or 2 points. While the study of graphs and voting games is usually associated with applied mathematics, the theory of topological spaces is counted among most fundamental theories of "pure" mathematics. A *topological* structure in a set E can be defined as a collection T of subsets of E – they are called *open sets* – which meets the following axioms: (1) The empty set $\varnothing$ and E are in T; (2) If $A\in T$ and $B\in T$, then $A\cap B\in T$ (the intersection of any two open sets is an open set); (3) The union of any collection of open sets is an open set.

That's how unbelievably simple are conceptual foundations of *topology* – the mathematical discipline that formalizes in the broadest way the notions of closeness and convergence. We say that an infinite sequence (e_n) of points of *E converges* to a *limit e* if in every open set containing point e there lie almost all terms of the sequence, where "almost all" means "all except at most a finite number." Any open set containing a point is called its *neighborhood*; intuitively, it is a set points which are "close" to a given point, however the degree of closeness may not be specified at this level of generality. If for any two distinct points there exist disjoint neighborhoods – a topological space with this property is called a *Hausdorff space* – any sequence may have no more than one limit.

Although the topology axioms should be intelligible to everyone familiar with elementary set-theoretic terminology, you may feel uneasy in a too abstract conceptual setting where intuition supported by sensual experience may fail. That's why those who, like Fararo (*Mathematical Sociology. An Introduction to Fundamentals*, 1973), embark on teaching modern mathematics to sociologists, begin from introducing "visible" topological spaces, such as the 2- or 3-dimensional Euclidean space, and stop at defining a general *metric space*, the species of structure that is constructed with the help of the set $\mathbb{R}$of real numbers.

A *metric* or *distance* on a set E is defined as a mapping ρ of $E\times E$ into $\mathbb{R}$ satisfying the following axioms: (1) For any $e,f\in E$, $\rho(e,f)=0$ if and only if $e=f$; (2) $\rho(e,f)=\rho(f,e)$, for any $e,f\in E$; (3) $\rho(e,f)\leq\rho(e,g)+\rho(g,f)$, for any $e,f,g\in E$. These axioms imply that the distance of any two points is greater than or equal to 0 ($0=\rho(e,e)\leq\rho(e,f)+\rho(f,e)= 2\rho(e,f)$, hence $\rho(e,f)\geq 0$).

The familiar example of a metric space is the set $\mathbb{R}^n$ of n-element sequences of real numbers with the *Euclidean distance*. For $x=(x_1,\dots,x_n)$ and $y=(y_1,\dots,y_n)$, it equals the square root of $\Sigma(x_i-y_i)^2$ and is interpreted geometrically as the length of the segment of the straight line joining points x and y. Then, for any three points in $\mathbb{R}^n$, the sum of the lengths of any two sides of the triangle with these points as vertices is greater than o equal to the lengths of the third side. This well known property of the Euclidean distance has become the third axiom of a more general theory, the theory of metric spaces, in which "closeness" is given a quantitative meaning. Two definitions link this theory to general topology. First, the *open ball* of radius $r>0$ centered at a point e is defined as the set of points which lie at distance smaller than r from e, symbolically, $B(e,r)=\{f\in E\colon \rho(e,f)<r\}$. Next, open sets are defined as the family of those subsets A of E which satisfy the condition: for every point a in A, there exists an $r>0$ such that $B(e,r)\subset A$. In particular, any open ball is an open set.

1.5.5. The set $\mathbb{R}$ is itself endowed with three interrelated structures: a relation written as $\leq$ (sharp inequality $r<r'$ is defined by the condition $r\leq r'$ and $r\neq r'$), and two algebraic operations: addition (+) and multiplication ($\cdot$). The axioms of the theory of real numbers are divided into five groups.

(1) The axioms of *continuous order* are the following: 1.1. For any $r,s\in\mathbb{R}$, if $r\leq s$ and $s\leq r$, then $r=s$; 1.2. For any $r,s,t\in\mathbb{R}$, if $r\leq s$ and $s\leq t$, then $r\leq t$; 1.3. For any $r,s\in\mathbb{R}$, $r\leq s$ or $s\leq r$; 1.4. (the axiom of continuity) For any nonempty subset S of $\mathbb{R}$, the set M(S) of upper bounds of S – it is the set of all $r\in\mathbb{R}$ such that $s\leq r$, for any $s\in S$ – is empty or contains the smallest element, or an element r_0 such that $r_0\leq r$, for any $r\in$M(S)); r_0 is called the *supremum* or least upper bound of S.

(2) *Addition* is characterized by four axioms which jointly state that $(\mathbb{R},+)$ is a commutative group.

(3) The next four axioms pertain to *multiplication* as an operation restricted to the set $\mathbb{R}-\{0\}$, where 0 stands for the neutral element for addition. It is postulated that $(\mathbb{R}-\{0\},\cdot)$ is a commutative group.

(4) The axiom connecting two operations in $\mathbb{R}$: $(r+s)\cdot t=r\cdot t+s\cdot t$, for any $r,s,t\in\mathbb{R}$.

(5) Two axioms which relate order to addition and multiplication: 5.1. For any $r,s,t\in\mathbb{R}$, if $s\leq t$, then $s+r\leq t+r$; 5.2. For any $r,s,t\in\mathbb{R}$, if $0\leq r$ and $s\leq t$, then $r\cdot s\leq r\cdot t$.

The set $\mathbb{N}$ of natural numbers can be treated as a subset of $\mathbb{R}$, the neutral elements of the groups $(\mathbb{R},+)$ and $(\mathbb{R}-\{0\},\cdot)$ being identified with natural numbers 0 and 1.

The axiomatics of real numbers is given here after the notes I wrote up, attending the course of mathematical analysis, or *la mathématique tout court*, by Professor Stanisław Łojasiewicz. The lecturer just mentioned Cantor and Dedekind's methods for demonstrating the existence of a mathematical object $(E,\leq,+,\cdot)$ satisfying the axioms – such an object is termed a *continuous ordered field* – yet he did not spare his students a detailed proof of a theorem that appears to every mathematical structuralist more important than a particular method of constructing real numbers. The theorem states that any two continuous ordered fields are isomorphic. "Uniqueness up to isomorphism" is required of a mathematical object in order that its base set could serve as auxiliary set in constructing complex species of structure.

The set $\mathbb{R}$, which is needed to define metric space, plays the auxiliary role in defining many other species of structure. A *real vector space* is a commutative group

$(E,+)$ with a mapping of $\mathbb{R}\times E$ into E. This additional structure – termed *multiplication* of *vectors* (elements of E) by *scalars* (real numbers) – is assumed to satisfy the following axioms: $\alpha(e+e')=\alpha e+\alpha e$, $(\alpha+\beta)e=\alpha e+\beta e$, $1e=e$. The expression $\alpha_1 e_1+\cdots+\alpha_n e_n$ is called a *linear combination* of vectors $e_1,\ldots,e_n$. The real vector space is said to be *finite-dimensional* if for some n there exists a collection of vectors $e_1,\ldots,e_n$ such that any $e\in E$ equals a linear combination of these vectors with some scalar coefficients $\alpha_1,\ldots,\alpha_n$. The minimum n for which such a representation of every vector is possible is called the *dimension* of the vector space E. The familiar example of an n-dimensional real vector space is the set $\mathbb{R}^n$ with the sum $x+y$ of $x=(x_1,\ldots x_n)$ and $y=(y_1,\ldots x_n)$ defined as $(x_1+y_1,\ldots,x_n+y_n)$, and αx as $(\alpha x_1,\ldots,\alpha x_n)$. The *length* of any vector $x\in\mathbb{R}^n$ is defined as the Euclidean distance of x from $0=(0,\ldots,0)$.

1.5.6. The notion of isomorphism turns out in a sense more fundamental than that of structure. While you can't define a given species of structures unless you specify a concrete construction procedure, often you have an option to do it in few ways which differ with typical characterization and/or axiomatics, yet in each case the same 1–1 mappings become isomorphisms. More formally, let Σ and T be two species of structure with the same base sets. Two *species of structure* are said to be *equivalent* if: (1) One can define a procedure applicable to any structure s of species Σ of which the output is a structure t of species T; in particular, the axioms of T must be deducible from the axioms of Σ after defining the terms used in the former by means of those occurring in the latter; (2) There is an analogous procedure for constructing a structure $s(t)$ of species Σ from a given structure t of species T; (3) The two procedures are inverse of each other, which means that by applying the second procedure to $t(s)$ we obtain again s, and so is for the composition of the two procedures in the opposite direction.

For example, the structure of an *undirected graph* with point set E can be defined equivalently as a symmetric relation $R\subset E\times E$ or as a family $L\subset\mathcal{P}(E)$ such that every element of L has 1 or 2 points. The set L of *lines* is obtained from R by replacing every ordered pair (e,f) in R with the subset $\{e,f\}$ of E. Since two *arcs* (e,f) and (f,e) in R yield the same *edge* in $L(R)$, the reverse procedure must assign to $\{e,f\}$ both (e,f) and (f,e) so that the relation $R(L)$ be symmetric. An undirected graph was defined earlier (1.2.6) in the second way. We added there the condition that L does not contain a *loop*, or an edge of the form $\{e\}$. Under the relational definition, this condition takes the form: R is *irreflexive*, that is, for any $e\in E$, $(e,e)\notin R$.

If Σ and T are equivalent, then any 1–1 mapping φ of E onto E' is an isomorphism of the objects (E,S) and (E',S') with Σ-structures if and only if φ is an isomorphism of the objects (E,T) and (E',T') with T-structures obtained from S and S'.

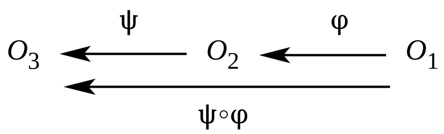
Can a definition of isomorphism be stated without explicit reference to the structures of the objects (E,S) and (E',S')? Yes, but one needs first to introduce the notion of "morphism" as a mapping of E into E'. Then an isomorphism is defined as a morphism having an inverse, the inverse of a morphism being a morphism whose compositions in either order with the given morphism yield the identity mappings i_E and $i_{E'}$. Morphisms appear in Bourbaki's work as special mappings that enable comparing mathematical objects having structures of the same species. This may mean, in particular, that one object is "immersed" in the other in the sense of being isomorphic with its "subobject" or one is the other's "image" preserving some "structural" characteristics of the "original."

Let $O_1=(E_1,S_1)$ and $O_2=(E_2,S_2)$ be two mathematical objects with structures of the same species. The *set of morphisms from O_1 to O_2* is defined as a certain subset of the set of all mappings of E_1 into E_2. In defining this subset, one has to make sure that two conditions be met: (1) the composition of two morphisms which can be composed with each other should be a morphism, that is, if φ is a morphism from O_1 to O_2 and ψ is a morphism from O_2 to $O_3=(E_3,S_3)$, then $\psi\circ\varphi$ should be a morphism from O_1 to O_3; (2) for any 1–1 mapping φ of E_1 onto E_2, φ is an isomorphism of O_1 and O_2 if and only if φ is a morphism from O_1 to O_2 and the inverse mapping φ^{-1} is a morphism from O_2 to O_1. As these conditions may be met by various classes of mappings, the notion of morphism, unlike that of isomorphism, is not always determined uniquely by the definition of a given species of structure. Hence, quite naturally, there emerged the idea that objects, morphisms and composition of morphisms be introduced as primitive terms of a theory that would be able to render more adequately the substantive unity as well as internal diversity of the mathematical universe.

1.5.7. Such a theory, known a the *category theory*, which was being developed parallel to Bourbaki's theory of structures, "has had – to quote two Polish mathematicians (Semadeni and Wiweger 1978: 13) – great influence on today's formulations of abstract mathematical theories and maybe we shall witness a breakthrough similar to that brought about by Cantor's ideas several decades ago." The conceptual apparatus of the category theory has also penetrated into applied mathematics, including social network analysis (Lorrain and White 1971). The cited paper inspired Fararo (1973: 564–569) to include in his book some information on categories.

A category $\mathcal{C}$ is defined by specifyng the *class* Obj($\mathcal{C}$) of its *objects* and the *class* Mor($\mathcal{C}$) of its *morphisms*, also called *arrows* for their graphical representation. Mor($\mathcal{C}$) is assumed to be the union of the *sets* $\mathrm{Mor}(O_1,O_2)$ over all ordered object pairs (O_1,O_2). The elements of $\mathrm{Mor}(O_1,O_2)$ are called *morphisms from O_1 to O_2*. For any two distinct ordered object pairs, the respective morphism sets are assumed to be disjoint, which implies that to any morphism there corresponds exactly one ordered pair of objects, referred to, respectively, as the *domain* and *codomain* of a given morphism.

The third primitive term of the category theory is the *composition* of morphisms. It is defined for any two morphisms φ and ψ such that $\varphi\in\mathrm{Mor}(O_1,O_2)$ and $\psi\in\mathrm{Mor}(O_2,O_3)$ as a morphism, noted $\psi\circ\varphi$, from O_1 to O_3.



The composition of morphisms is assumed to satisfy the axiom of *associativity*, which property is defined similarly as for mappings or semigroup operations. The last axiom states that for any object O there exists a morphism ι_O from O to O such that

$\psi\circ\iota_O=\psi$, for any morphim ψ with domain *O*, and $\iota_O\circ\varphi=\varphi$, for any morphism φ with codomain *O*.

The classical Zermelo-Fraenkel set theory may not suffice as ontological basis of the category theory as the latter must handle "larger" entities than ordinary sets. Since an attempt to speak of the *set* of all sets leads to a contradiction, in order to accept the existence of a collective entity which consists of *all* sets, one has to resort to a richer mathematical ontology that starts from the notion of a *class* and distinguishes between *sets* and *proper classes*, or classes that are not sets.

However the category theory absorbed most of results of Bourbaki's theory of structures, it did not invalidate the latter. A bridge between the two theories is the concept of a *concrete category*, or a category in which every object is built over a set, called its *base set*, and any morphism from O_1 to O_2 takes the form (O_1,φ,O_2), where φ is a mapping of the base set E_1 of O_1 into the base set E_2 of O_2. Concrete are all categories whose objects are sets endowed with structures of the same species as well as the *category of sets* with "bare" sets as objects and all mappings as morphisms.

A shift of emphasis from objects to transformations is not the only result of the impact of the category theory on all mathematics. It is even more important that any category can be considered as macroobject which can be "transformed" into another macroobject of sort. To make inter-category comparisons possible, one only needs to define a counterpart of morphism. A *covariant functor* from a category $\mathcal{C}$ to a category $\mathcal{C}'$ is defined as a pair of mappings (in a generalized meaning) both noted with the same symbol Φ. The first of them assigns to any object *O* in $\mathcal{C}$ the object $\Phi(O)$ in $\mathcal{C}'$, while the second transforms Mor($\mathcal{C}$) into Mor($\mathcal{C}'$) in such a way that: (1) the image $\Phi(\varphi)$ of any morphism φ from O_1 to O_2 is a morphism from $\Phi(O_1)$ to $\Phi(O_2)$; (2) the composition of morphisms is preserved by Φ; (3) $\Phi(\iota_O)=\iota_{\Phi(O)}$.

1.5.8. Our inquiry has reached the point when we are in a position to clarify the meaning of mathematical structuralism. On all three levels of analysis, that is, the intra-object level, the inter-object or intra-category level, and the inter-category level, mathematics deals with structural or invariant properties, intuitively, those properties that do not depend on particular intrinsic nature of entities studied at a given level. Recall that properties have been identified (see 1.2.1) with variables taking 1 and 0 as the only values telling us, respectively, that a unit of analysis (which on the intermediate level is a mathematical object) does or does not have a given property.

Let *V* be a variable which assigns numerical values to the elements of some subclass $\mathcal{D}$ of Obj($\mathcal{C}$). *Structural variables*, also called *structural parameters* or *invariants*, are defined by the condition: $V(O_1)=V(O_2)$, for any two isomorphic objects O_1 and O_2 in $\mathcal{D}$. On the inter-category level, structural variables are defined as those whose values are preserved by special functors called *bijectors* or category isomorphisms. I will skip a more detailed explanation to avoid getting involved in more "metatheory" than needed by ordinary mathematicians whose research hardly ever goes beyond the intra-category level or even the lower level, or that of a fixed set (often called a "space") endowed with one or few interrelated structures, $\mathbb{R}^n$ with metric and vector structure being a familiar example.

On that lowest level, structural analysis consists in the study of structural variables defined on the set of "parts" of a given mathematical object. "Structural" means now "preserved by the automorphism group acting on the set of parts." To give a precise meaning to this statement, we must define the term "action of a group on a set." Before we do it, we show few examples of global structural variables. The qualifiers "global" and "local" will be used here to mark the distinction between variables defined on the intra-category (inter-object) level and those defined on the intra-object level. Some structural global variables, such as the number of elements of the base set E of an object $O=(E,S)$, are defined similarly in any concrete category, more exactly, in its subcategory made up of all objects with finite base sets. If a set A is *finite*, that is, if A has as many elements as the set $\{1,\ldots,n\}$, for some $n\in\mathbb{N}$, we will use the symbol $|A|$ to denote *cardinality* (number of elements) of A. Any isomorphism of the objects $O=(E,S)$ and $O'=(E',S')$ is 1–1 mapping of E onto E'. Thus, if E and E' are finite sets, we have $|E|=|E'|$.

A "more structural" (dependent on S) example of a universally applicable global structural parameter is the number of automorphisms of O. To prove that the condition defining structural variable is met, notice that if objects O and O' are isomorphic, so are their symmetry groups Aut(O) and Aut(O'). If E and E' are finite sets, Aut(O) and Aut(O') are also finite and $|\mathrm{Aut}(O)|=|\mathrm{Aut}(O')|$.

Many global structural variables are specific to particular categories. *Dimension* is a global structural parameter in the category of finite-dimensional real vector spaces. Moreover, any two real vector spaces with the same value of this parameter are isomorphic; hence, every n-dimensional real vector space is isomorphic with $\mathbb{R}^n$. Thus, dimension is an example of a *complete invariant*.

In the category of directed graphs, the simplest global structural variables are the number of p points and the number q of arcs of a digraph. These two structural parameters do not form a complete set of invariants because, for most values of p and q, there are many nonisomorphic digraphs (E,R) such that $|E|=p$ and $|R|=q$.

1.5.9. There exist categories in which any two objects are isomorphic. The category of continuous ordered fields is the example we have already discussed (1.5.5). Such a category, as it is often said, has "one object up to isomorphism." In most concrete categories, however, Obj(C) consists of infinitely many nonisomorphic objects. In a concrete category, any two objects with finite base sets E and E' can't be isomorphic if $|E_1|\neq|E_2|$. Hence, we can only try to count equivalence classes for the restriction of the isomorphism relation to the set $\mathrm{Obj}_E(\mathsf{C})$ of all category objects with the same finite base set E. For convenience, let me refer to the elements of this set as *configurations*. In my old paper (Sozański 1992), I proposed *structural classification* as the term to denote the partition of $\mathrm{Obj}_E(\mathsf{C})$ into isomorphism classes.

Fararo (1973: 122) applied the same construction – very general, as it were – to the set $\{(E,R): R\subset E\times E)$ of all directed graphs having E as their common set of points. His decision to term "structures" the isomorphism classes of "relational systems" looks like an attempt to formalize Nadel's concept of "structure as form" (1.4.1). I think it's better to stay with the understanding of structure (in particular, a binary relation) as a thing *constructed* from things – from points of a fixed set. When two objects obtained by endowing a set E with a structure of the same species are isomorphic, I will say that they have the "same structural form" or are "structurally similar" instead of saying that they have the "same structure". Thus, the sets which make up a structural classification will not be called “structures” but *structural forms*. My second terminological choice is to see in relations a special kind of structures rather than reserve the term “structure” for relations only.

We illustrate the notion of structural classification using as configurations all 3-person voting games (see 1.5.5). Let $E=\{1,2,3\}$ denote the assembly of voters. Since $|E|=3$, we have $|\mathcal{P}(E)|=2^{|E|}=2^3=8$ and similarly, $|\mathcal{P}(\mathcal{P}(E))|=2^{|\mathcal{P}(E)|}=2^8=256$. It is not difficult to verify that only 11 out of 256 3-point hypergraphs satisfy the voting game axioms given in 1.5.4. These 11 configurations fall under 5 isomorphism classes. They formally describe all “structurally distinct rules” (recall an equivalent formalization given in 1.3.8) any 3-person group can use for decision-making.

Structural form (1) is a formal model of the *consensus* rule. To decide on any issue, all members of the assembly must vote unanimously. Formally, the voting game (E,W_1), the unique element of form (1), has E as the only winning coalition, that is, $W_1=\{E\}$. Form (2) is also represented by a single voting game with $W_2=\{\{1,2\},\}\{1,3\},\{2,3\},E\}$. It is known as the *simple majority* rule. Form (3) consists of 3 *dictatorial* decision systems with $W_3=\{\{1\},\{1,2\},\{1,3\},E\}$, $W_4=\{\{2\},\{1,2\},\{2,3\},E\}$, $W_5=\{\{1\},\{1,2\},\{1,3\},E\}$. These three isomorphic voting games differ with person occupying the dictatorial position (i is called a *dictator* if $\{i\}$ is a winning coalition). 3 *duumvirate* systems with $W_6=\{\{1,2\},E\}$, $W_7=\{\{1,3\},E\}$, $W_8=\{\{1,3\},E\}$ form structural form (4). In $W_9=\{\{1,2\}, \{1,3\},E\}$, voter 1 is the only group member who is in a position to veto any decision: no winning coalition can form without his consent. The other 2 configurations of form (5) are: $W_{10}=\{\{1,2\},\{2,3\},E\}$, $W_{11}=\{\{1,3\},\{2,3\},E\}$, with actor 2 or 3 in the *vetoer* position.

1.5.10. It is usually much easier to calculate the number of configurations than that of structural forms. The second problem can often be solved by applying a formula that belongs to the theory of “action of a group Γ on a set X.” The theory's combinatorial implications, much deeper than the results we recall below (after Lang 1970), are due to Pólya (the source text usually referred to is De Bruijn 1964).

The *action of a group* Γ *on a set* X is defined as a mapping of $\Gamma\times X$ into X that assigns to any ordered pair (γ,x) an element of X, written γx, and satisfies the following axioms:

(1) $\iota x=x$, for any $x\in X$, where ι is the neutral element of Γ;
(2) $(\gamma\circ\gamma')x=\gamma\gamma'x$, for any γ, $\gamma'\in\Gamma$, $x\in X$, where $\circ$ stands for the operation in Γ.

The action of Γ on X is used to define the following relation on X:

$x\sim x' \Leftrightarrow$(df) $x'=\gamma x$, for some $\gamma\in\Gamma$.

Its reflexivity and transitivity follows from the axioms of group action. To prove that the relation is symmetric, being therefore an equivalence, suppose that $x\sim x'$. By applying γ^{-1} to both sides of $x'=\gamma x$, we get $\gamma^{-1}x'=\gamma^{-1}\gamma x$. Since the right hand side equals $(\gamma^{-1}\circ\gamma)x=\iota x=x$, we conclude that $x=\gamma^{-1}x'$, that is, $x'\sim x$.

The equivalence classes of ~ are called *orbits* of elements of X. The orbit of x is the set of the form $\mathrm{Or}(x)=\{\gamma x: \gamma\in\Gamma\}$. Note that $\mathrm{Or}(x)=\mathrm{Or}(x')$ if and only if $x\sim x'$. The partition of X into orbits induced by the action of Γ on X will be noted X/Γ.

For any two elements x and x' of X, let $\Gamma_{xx'}=\{\gamma\in\Gamma: \gamma x = x'\}$. The set $\mathrm{St}(x)=\Gamma_{xx}$, is a subgroup of Γ, which means that $\mathrm{St}(x)$ is a non-empty subset of Γ ($\iota\in\mathrm{St}(x)$) and for any $\gamma,\gamma' \in St(x)$, $\gamma\circ\gamma' \in St(x)$ and $\gamma^{-1} \in St(x)$. St(x) is termed the *stabilizer* of x. If Γ and X are finite sets, then, for any $x\in X$, the following formula holds true

$$|\Gamma| = |Or(x)| \cdot |St(x)|$$

The proof begins from the observation that Γ is the union of its pairwise disjoint subsets of the form $\Gamma_{xx'}$, where x is a fixed element of X and x' runs over the orbit of x. Hence, for any x, $|\Gamma|$ equals the sum of $|\Gamma_{xx'}|$ with $x' \in Or(x)$. The next step is to show that if $x' \in Or(x)$, then $|\Gamma_{xx'}| = |\Gamma_{xx}|$. If $x' \in Or(x)$, then, for some $\gamma' \in \Gamma$, $x' = \gamma' x$. By assigning $\gamma'\circ\gamma$ to any γ in Γ_{xx}, we get a 1–1 mapping of Γ_{xx} onto $\Gamma_{xx'}$.

For any $\gamma\in\Gamma$, let X_γ stand for the set of those points of X that are not "moved" by γ, that is, $X_\gamma = \{x\in X: \gamma x = x\}$. The set of all ordered pairs (γ,x) such that $\gamma\in\Gamma$, $x\in X$, and $\gamma x = x$ can be written either as $\{(\gamma,x): \gamma\in\Gamma, x\in X_\gamma\}$ or as $\{(\gamma,x): x\in X, \gamma\in St(x)\}$. Hence, the number of its elements can be expressed as the sum of $|X_\gamma|$ over all γ in Γ or the sum of $|St(x)|$ over all x in X. Since $|St(x)|=|\Gamma|/|Or(x)|$, the second sum equals $|\Gamma|\,|X/\Gamma|$. On equating the two sums, we arrive at the formula

$$|X/\Gamma| = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} |X_\gamma|$$

known as Burnside's lemma. Thus, to compute the number of orbits one only needs to determine, for any $\gamma\in\Gamma$, the number of solutions of the equation $\gamma x = x$. Note that the solutions are "fixed points" of the mapping $T_\gamma: X\to X$, where $T_\gamma(x) = \gamma x$, for any $x\in X$. My papers (Sozański 1980, 1992) on "social combinatorics" show few examples when the calculation task is feasible.

1.5.11. Let us take now as Γ the group of permutations of a set E and as X the set of configurations based on a finite set E. Let the result of action of a permutation φ on a configuration (E,S) be the configuration $(E,\varphi(S))$, where $\varphi(S))$ is the structure transported from the object (E,S) to the set E through φ. It is easy to notice that the stabilizer of a configuration is its symmetry group and the formula $|\Gamma| = |Or(x)|\,|St(x)|$ takes the form $n! = |Aut(O)|\,|Or(O)|$. Hence, the size of the orbit of an object varies inversely with the size, ranging from 1 to $n!$, of its symmetry group. If $|Aut(O)| = n!$ (every permutation of E is an automorphism of O), then $|Or(O)| = 1$ (O is the only element of the orbit of O).

Since the orbits coincide with structural forms, Burnside's formula can be employed to count them. The number of structural forms increases very fast with the cardinality of the base set. Hence, the structural classification is usually a too fine partition to be practically useful. Configurations and structural forms are therefore further divided into broader sets corresponding to the values of some structural parameters, including those having an "empirical meaning." Combinatorial issues associated with the mathematical notion of structure seem at first sight far from empirical applications. However, many problems had their origins outside pure mathematics, to mention just one classical example: the problem of determining the number of isomers of a chemical compound.

1.5.12. We shall close our panoramic overview of the mathematical universe with showing how structural analysis works on the level of a fixed mathematical object $O=(E,S)$. Analysis of any thing consists in discerning in it certain parts which can be compared with one another with respect to various properties. Which properties shall we call "structural"? We can take as parts of O the elements of the base set E, its subsets or structured subobjects with structures "inherited" from O. Let X_O denote the set of parts of a given type. Having defined the action of the automorphism group Aut(O) on X_O, we define *local structural variables* as those local variables that are constant on the orbits generated by the action of Aut(O) on X_O.

In the case of X_O=E, the action of Aut(O) on E is defined by the formula $\alpha e=\alpha(e)$, for any $\alpha\in$Aut(O) and any $e\in E$. To give an example of a local structural parameter, consider a hypergraph (E,H) and assign to any $e\in E$ the number of edges in H containing e. This parameter, when applied to a graph, is called the *degree* of a point (see more in 1.2.6). For a voting game, it provides us with a simple measure of voting power. The more winning coalitions an actor is a member of, the greater his voting power.

The *outdegree* of a point e in a directed graph (E,R) is defined as the number of arcs having e as the common initial point, symbolically, od(e)=$|\{f\in E: (e,f)\in R\}|$. The *indegree* of e is defined similarly: id(e)= $|\{f\in E: (f,e)\in R\}|$. These two local structural parameters are probably best known to sociologists because they have been used since 1930s in theory and research on small groups. When they were discovered independently by Moreno and his collaborators, the mathematicians, who had by then considered graph theory as a minor chapter of algebraic topology, were not yet ready for cooperation with sociologists. Since the publication of *Structural Models* by Harary, Norman, and Cartwright (1965), the exchange of ideas has become regular practice, social scientists being more and more active side, to mention only Markovsky, Willer, and Patton (1988) who invented *Graph-theoretic Power Index*. It is a local structural parameter designed for use in graphs modeling the communication structure in network exchange systems.

The action of Aut(O) on X_O=$\mathcal{P}(E)$ is defined by the formula $\alpha A=\alpha(A)$, for any $A\subset E$. Let O be a set E with a metric ρ. A subset A of E is said to be *bounded* if there is a number r>0 such that $\rho(a,a')\leq r$, for any $a,a'\in A$. Boundedness is a structural property of the subsets of a metric space. To make this statement meaningful and prove it, we need to know what is *isometry*, or isomorphism of metric spaces (E,ρ) and (E',ρ'). It is a 1–1 mapping φ of E onto E' such that $\rho'(\varphi(e),\varphi(e')=\rho(e,e')$. If A is bounded and $\alpha\in$Aut(O), then $\alpha(A)$ is also bounded because $\rho(\alpha(a),\alpha(a'))=\rho(a,a')\leq r$.

The so called Erlangen program, which is recognized as the first conscious manifestation of structuralism in mathematics, specified that the objective of geometry should be the study of invariants of various transformation groups acting on the subsets of the Euclidean space or other spaces. In the social sciences, one can rarely find comparable specificity in formulating the principles of structural analysis, albeit almost all theorists have something to say or feel obliged to say something about "structure." The mathematicians seldom speak of "structure" in general. They study relations, operations, and topologies often without being aware that they deal with various kinds of structure.

Every mathematical discipline reaches the stage of conscious structuralism with defining isomorphism for the mathematical objects to be studied. New disciplines may need some time to develop their ways of structural analysis. When I worked on a special kind of games called “minimal social situations” (Sozański 1992), I could have listed just one structuralist paper (Rapoport and Guyer 1965), the one concerning structural classification of similar configurations: “2×2 matrix games with ranked payoffs.”

1.6. Mathematical modeling of empirical systems

1.6.1. It is clear that the statements like Nadel's definition of structure suggest that the mathematical understanding of structure is not alien to the social sciences. When you reflect on how to implant mathematical structuralism in an empirical science, the first idea that springs to mind is to define the subject matter of the given discipline as a “category” of “structured” empirical objects. The strategy of finding empirical *analogues* of mathematical concepts, which reminds the second variety of heuristic naturalism (see 1.3.20), requires, first of all, that a notion of “structural similarity” be defined for real-world complex wholes. Such an empirical counterpart of isomorphism – a notion that is needed to define structural variables – would differ from isomorphism defined in mathematics in that it would not completely abstract from the stuff which empirical entities are made of.

An alternative approach is based on *modeling* empirical objects by mathematical objects. Structure and isomorphism, albeit they indirectly refer to empirical objects, retain then their strict mathematical meaning because they pertain to mathematical objects that are models of empirical objects.

Minimal *syntactic* codification of any scientific language consists in pointing out specific terms and contexts, considered meaningful, in which these terms occur together with nonspecific (logical and mathematical) terms and possibly colloquial expressions which cannot be dispensed with even in mathematics. At the next step, a formal language is created by stating explicit rules for producing well-formed statements. The language of any *mathematical* theory not only admits of complete syntactic formalization, but it “forces” for itself a semantic interpretation under which “things” that correspond to “signs” are set-theoretic constructs. As I have already argued, sets made up of people do exist in the real social world. Often they are quite concrete tangible entities, like my 3-person nuclear family (see 1.3.21). If so, why not to use set-theoretic ontology to codify the semantics the language of any *empirical* discipline as well? Although a mathematical object is constructed primarily so as to get an interpretation of a formal language, it may stand at the same time in a *modeling relation* with some empirical objects. If an empirical object is conceived as if it were itself a mathematical object, say, if a social group is equated with a set of people endowed with a binary relation in the mathematical meaning of term, then the correspondence between the model (made up of more abstract material, say, natural numbers that stand for group members) and the modeled object becomes a regular isomorphism – if the model reflects “faithfully” the “nature” of the empirical object, or as a morphism if the model in a sense simplifies the reality.

Nevertheless, one must not confuse the subject matter of *logical semantics*, the branch of logic that deals with set-theoretic interpretations of formal languages with that of *formal methodology of empirical sciences*. The relationship between nonmathematical reality and its mathematical representations may not be completely described in a purely logical metalanguage. The world of experience and the world of mathematics have, to be sure, more in common than it appears to many simple or refined minds, yet one must not forget that a social group differs in its mode of existence and properties from a set of real human beings or even a relational system whose elementary components come from the world out there. Similarly, the physical space which contains our bodies is more mysterious than $\mathbb{R}^3$, or its mathematical representation obtained by introducing a coordinate system. Material objects mentioned by Nadel ("crystal, wooden block, or soup cube"; see 1.4.1) do not resemble in many respects the subset of $\mathbb{R}^3$ which is their geometric model (see 1.3.21). Yet "what can be said" about their common *form* "can be said clearly" only in the language of the mathematical discipline that deals with $\mathbb{R}^3$ and its subsets.

Inventing mathematical models engages both creative abstract thinking and physical contact with a piece of empirical reality. First, a mathematical representation must be thought up to formally describe the shape of a material object. Next, the values of some empirical variables need to be *measured* to assess adequacy of the chosen representation as well as to obtain a *parametric description* of the object. "Describing the appearance of an object, or giving its measurements" appears in Wittgenstein's list of language games just past "giving orders, and obeying them" (see 1.3.18). Mathematical modeling of empirical systems is the best strategy in the epistemic language game – insofar as the players' aim is to produce objective knowledge of the reality they are going to study.

1.6.2. A mathematical model is recognized as empirically valid if it enables formulating and testing hypotheses on certain properties or behavior of the modeled class of empirical systems. To state a *hypothesis*, one needs to define first a set of *variables* (see 1.2.1). It is their nature that is taken into consideration, above of all, in metatheoretical analyses of *formalized empirical theories* and *paradigms*. Which paradigms and theories shall be called *structural*? I propose a simple criterion: it is the use of *structural variables* – alone or along with nonstructural variables. It follows from the definition of the attribute "structural" as "preserved by isomorphism" – with the term "isomorphism" borrowed from the basic mathematical glossary – that "structural approach" in empirical sciences acquires a definite meaning as soon as a mathematical representation has been devised for a given class of empirical systems. Both structural and nonstructural variables describe differences between empirical objects, but in the case of structural variables a numerical value can be assigned to an empirical object only via its mathematical model.

While such an understanding of "structuralism" may appear too narrow – for insistence on a more "technical" explication of the notions of "structure" and "isomorphism" – it is in fact very inclusive, even more so than Wellman's network "structural analysis" (see 1.4.8), as it provides for considering non-relational structures and encompasses structural paradigms that differ with interpretations of "structure" as "constraint on a process" or "pattern produced by a process." The latter distinction (see 1.4.10) cannot be expressed in the language of Bourbaki's "general theory of structures" (outlined in see Section 1.5 with introductory presentation of few key "species of structure"). We can't expect from mathematics to provide us with precise reformulations of all issues that have been raised within structuralist metatheorizing in the social sciences. Nevertheless, a seemingly philosophical problem of how the concepts of "structure" and "form" are related to each other can be solved with the use of relatively simple mathematical tools. The solution (see 1.5.9) I proposed in my 1992 paper helps remove some uncertainty about the meaning of these concepts we find in informal discourse of old "formal sociology" (Simmel, Nadel) as well as in quite formal statements of mathematical sociologists (Fararo).

Many sociologists too little know, misunderstand or dismiss mathematically-inspired structuralism as allegedly unsuitable for the social sciences. There are exceptions, however, as can be seen from the following quotation from William Goode's contribution (Goode 1975: 74) to the *Approaches to the Study of Social Structure* (Blau 1975 Ed.).

> "Some pieces of physical, biological and social reality, I believe, do approximate more closely to genuine structures in the strict sense that the *arrangement of the parts* controls much of the variance in the phenomena. Wittgenstein said in this connection that we should pay attention to the network, the geometry of its arrangement, and not to the characteristics of the things the net describes; if a field has so progressed that it can create such an intellectual structure, perhaps that advice is wise. However, it is not true that an examination of all social relations, all biological phenomena, or all cultural patterns will easily disclose an underlying structure in which the arrangement of the structure is the most central set of variables to be considered."

Structural variables – those that refer to the "geometry" of the "arrangement of the parts" – play in this paradigm the role of independent variables that are believed to "control much of the variance" of nonstructural dependent variables. Goode remarks that an attempt to explain in this way "the variance in the phenomena" does not always need to be successful, but "a moment's thought will inform us that there are some areas of social behavior which might be good candidates for this kind of analysis" (1975: 74). The first example he recalled to support his claim was the use of structural properties of communication networks to account for differences in efficiency between task groups. This example has already been analyzed in this chapter (see 1.4.10).

1.6.3. Inventing general structural sociological paradigms has always been a favorite business of "grand theorists," but their books offer at best "structural insights." To make sure that structural approach is a workable paradigm in the social sciences, you should rather analyze seminal research papers. The structural paradigm that underlies Bavelas' theory and research defines the class of empirical objects to be studied as *task groups* – a kind of *social interaction system* endowed with a fixed *communication network.* The latter is interpreted as *constraint on actions* of group members and mathematically modeled by a connected undirected graph. In experimental research, *physical* channels (slots in the dividers separating cubicles for the actors) were used to construct a communication network in the laboratory setting. The same structure-constraint could have been enforced by instructing experimental subjects on who is and who is not permitted to communicate with whom, that is, by enforcing a *social* norm vulnerable to violation but assumed to be respected. The mathematical representation of the communication network is an integral part of the paradigm, while the nature of network ties is a secondary factor. As regards the network nodes (points of the graph), they represent positions in the social interaction system or actors that occupy them. To predict the actors' behavior as well as the behavior of the system as a whole – under various forms of structure-constraint – one must make certain assumptions on the actors' motivation (see 1.3.12 and 1.4.10). In task groups, the subjects are told by the experimenter to be oriented toward the achievement of the group goal, same for all groups. Once the groups do not differ in motivation, all independent variables in this paradigm are structural. The dependent variables, which are all nonstructural, are measures of group performance (the amount of time a group needs to complete the task) and satisfaction (the average of ratings made by group members).

The heuristic function of any structural paradigm of the kind – let us call it *inter-system structural analysis* – consists in pointing to structural variables as those among which might be the variables responsible for the differences between empirical objects with respect to definite dependent variables. The purpose of any specific *structural theory* falling under such a paradigm is to pick a concrete structural variable as *explanans* for the given nonstructural *explanandum*. In Bavelas' theory, it was a structural variable measuring centralization of the communication network that was suspected to have a strong effect on the level of group performance. To test this hypothesis, 4 out of 21 nonisomorphic connected 5-node undirected graphs were selected for the experiment. As Flament (1963: 50–52) noticed, Bavelas' graphs (3 trees and the 5-node cycle) also differed with the number of automorphisms, a global structural variable being another theoretically plausible explanans under the same paradigm.

The structural paradigm for the study of *network exchange systems* – social action systems in which actors negotiate with each other and conclude bilateral transactions – shares some features with the paradigm that guided the first research on task groups working under network constraint. While all members of a task group cooperate in pairs with the aim to contribute to the collective success, all actors in a network exchange system pursue their individual goals defined for each of them as earning as much as possible for himself. The two types of social systems differ therefore with the nature of actors' co-action and motivation. However, in both cases the range a dyads that are permitted to interact is determined by the same structure-constraint, a communication network modeled by an undirected graph.

1.6.4. The mathematical model of an exchange network contains an additional structure-constraint that forces a kind of interdependence of dyadic subsystems. The structure in question – later it will be called "exchange regime" – already appears in the example I have shown earlier in this chapter (see 1.2.6) to introduce the reader to the main topic of the next chapter: the general paradigm for the study of network exchange systems. I invoke again this example to illustrate how *local structural variables* (see 1.5.12) can be used to predict the values a given nonstructural variable for the same elements (the parts of a fixed system). Many structural network exchange theories – they will be analyzed in Part II – can be subsumed under this paradigm – *intra-system structural analysis* – which is a way to formalize the common sociological custom of explaining variance in income, attitudes, voting behavior, etc. by different "places" individuals occupy in a hierarchical organization or in any social system in which the actors' "positions" are comparable in terms of "structural similarity."

The paradigms in which *local* or *global* structural variables are used as independent variables are not the only structural paradigms. Structural variables may also play the role of dependent variables, for instance, in paradigms that relate some structural properties of a social system to some nonstructural properties of its human substratum, say, homogeneity/heterogeneity of the set of group members.

Still other structural paradigms contain structural variables only. The simplest "purely structural" paradigm may have just one structural variable assigning values to certain *configurations* (in the special meaning given to this term in 1.5.9). On assuming that all configurations are equally likely, one can compute the probabilities for all values of a given structural variable. These probabilities can be compared with relative frequencies determined from a set of configurations representing a sample of empirical objects. If the empirical distribution significantly departs from the random distribution, we say that a *structural bias* takes place. To give a simple example, assume that every member of a group made up of n_1 boys and n_2 girls is asked to name exactly one co-member of the opposite sex he or she likes most. The configurations so obtained are special directed graphs; their number equals $n_1^{n2}n_2^{n1}$. Consider the variable V which assigns to any configuration the number of pairs with mutual choice. V is the sum 0-1 variables of the form V_{ij}, i=1,...,n_1, j=1,...,n_2, where V_{ij} takes the value of 1 for every digraph in which ith boy chooses jth girl and conversely (these variables, unlike their sum, are not structural). Since the probability of a mutual choice for each pair equals $n_1^{-1}n_2^{-1}$, the expected valued of the *random variable* $V=\Sigma V_{ij}$ is equal to 1 (it is the sum of expected values of n_1n_2 0-1 variables). Hence, to ascertain whether interpersonal preferences in mixed-sex groups reveal a tendency toward reciprocity, one must compute the mean value of V from a sample of groups and verify if it significantly exceeds 1.

1.6.5. If the configurations having definite structural forms are observed with a much greater frequency than that predicted on the basis of a simple combinatorial analysis, you can try to explain this regularity by means of a deterministic or probabilistic mechanism working in a given empirical domain. A predominance of certain structural forms in a sample of empirical objects may also be interpreted as a result of stepwise transforming the initial structure until it attains its final shape marked by a structural bias.

The construction of a dynamical model for an empirical object begins from distinguishing its unvarying base and varying states. A *state* is usually identified with the sequence of current values of certain quantitative or qualitative variables (see also 1.4.6). If you ask an economist the current state of the Irish economy, she will probably let you know the values main economic indicators assumed by the end of the last year or month. A sociologist would answer a similar question concerning the state of the Polish society with a "discursive nonmathematical" narrative, as his discipline still suffers from the lack of standard parameters, especially those suitable for macrosocial objects. Old masters of "social theory" did little in this matter apart from Durkheim's idea to use the suicide rate as a social indicator. In microsociological studies, the unvarying base of a small group is the set of its members, while the states are all binary relations in it, including the empty relation, which is the initial state of a group formed from the individuals who have not known one another before their first meeting. In Newcomb's (1961) research on the "acquaintance process," the successive relations were determined by having the group members to answer sociometric questions every week. This example is mentioned here to show that a formally defined structure (in particular, a relation) may be interpreted not only as constraint or pattern but as a variable state – against the mental habit of seeing in any structure something firm and constant.

As regards the most mysterious object of human reflection, the concept of *time*, mathematics has offered to the empirical sciences two formal representations to be used in dynamical modeling. Under the *continuous time model*, when we say that a *dynamical system* – an object able to change its state – is *at time t in a state x*, we mean by t a *real number* – an element of the set $\mathbb{R}$ (see 1.5.5). Consequently, a *process* is represented as a mapping $t \to x(t)$ with t varying over an interval and $x(t)$ standing for the system's state at t.

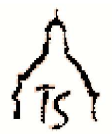
The *discrete time model* replaces $\mathbb{R}$ with the set of *natural numbers* $\mathbb{N}$. A process is then defined as an infinite sequence of states $x_0, x_1, \ldots$. The sequence beginning from the *initial state* x_0 may reach a *final state*, or a state x_n such that $x_{n+1}=x_n$, $x_{n+2}=x_n$, and so on (the system, having attained the state x_n, will stay in it forever). In a *cyclical process*, a fixed finite sequence of states returns infinitely many times (e.g. sunrise-sunset-sunrise...). That's all what can be said at this level of generality. If the system's *state space* X is endowed with a Hausdorff topological structure (see 1.5.4), then still another regular pattern is possible for an infinite sequence of states (x_i), namely, the sequence may be convergent to a limit x^*. The system may never attain the state x^*, but its current state is getting closer to x^* with each successive step of the process.

The objective of specific mathematical theories concerning the behavior of a dynamical system with a state space X is not to foretell the actual sequence of events, as it will unfold in real time before the eyes of an observer, but rather to determine all potential processes starting from all possible initial states. A simple deterministic theory, which uses the discrete time model and satisfies the above requirement, assumes that the transition from the current state to the next state comes about according to the same functional law independently of the stage and initial state of a process. Formally, if the system is in a state x in a moment k, then it will be in the state $F(x)$ in the moment $k+1$, where F is a transformation of X into X. As a consequence, the process which begins from a state x_0 has the form: $x_k=F^k(x_0)$, for $k=0,1,\ldots$ where $F^0(x_0)=x_0$ and $F^k(x_0)= F(F^{k-1}(x_0))$, so that the initial state determines uniquely all subsequent states.

Assume now that X is a Hausdorff space and F is a *continuous mapping*, which means, by definition, that for any $x \in X$ and for any neighborhood of $F(x)$ (in a metric space, it can be an open ball $B(F(x),\varepsilon)$ with $\varepsilon>0$), there exists a neighborhood of x ($B(x,\delta)$ with some $\delta>0$) such that F transforms its points into points in the neighborhood of $F(x)$ (if $\rho(x',x)<\delta$, then $\rho(F(x'),F(x))<\varepsilon$, where ρ is the distance in X). Any point x^* in X is called an *equilibrium state* if there exists an initial state x_0 such that x^* is the limit of the process $x_k=F^k(x_0)$. If a sequence (y_k) of elements of X, converges to y^*, then the sequence $(F(y_k))$ converges to $F(y^*)$. On applying this property of any continuous mapping to the sequence (x_k), we arrive at the formula $F(x^*)=x^*$, which means that equilibrium states coincide with *fixed points* of F. If a state of equilibrium is attained or taken as the initial state, then a change of state can be brought about only by an external intervention. When this happens to the system, it is interesting to know whether it will move toward the previous state by virtue of its internal dynamics. Hence the following definition of *stability* (see also the first appearance of this concept in 1.2.4): an equilibrium state is *stable* if for any point x in some neighborhood of x^*, the sequence $(F^k(x))$ converges to x^*.

Theorems on the existence of fixed points for various transformations help solve many specific problems in various branches of pure and applied mathematics. In Chapter 4 of this book (see also Sozański 1997), the Brouwer fixed point theorem will be used to prove the existence of an equilibrium state in a special dynamical system which is to formally represent how the actors, negotiating in a network exchange system, modify their current offers.

An introduction to the theory of dynamical systems has been the last of several of strictly mathematical themes intermixed in this long chapter with a good deal of general methodology and reflections on the foundations of social science, social mathematics, mathematical sociology, and structuralism. Trying to open for the sociologists a window on the world of mathematics, I could not avoid technical discourse in some places (e.g., 1.5.10), which could baffle those readers who are used to see in symbolic notation an obstacle rather than an aid in understanding key ideas. I selected for presentation those fragments of the "mathematical edifice" which in my opinion are most important from a metatheoretical vantage point or will appear later in this book. Chapter 2 will also be dominated by mathematical stuff, mostly graph theory, where lie the conceptual and theoretical foundations of the mathematics of exchange networks.



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